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29

A High Resolution Algorithm for Detection-estimation of Narrow-band Signals using Sensor Arrays without Eigendecompositions of Data Correlations

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A HIGH RESOLUTION ALGORITHM FOR DETECTION-ESTIMATION OF NARROW-BAND SIGNALS USING SENSOR ARRAYS WITHOUT EIGENDECOMPOSITIONS OF DATA CORRELATIONS

Technical report

by

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ABSTRACT

High resolution algorithms in sensor arrays lead to accurate results but with expensive eigendecompositions making its use in real-time applications such as mobile communications relatively difficult. In this technical report, a trade-off between accuracy and computational load is accomplished through a simplified algorithm which instead of eigendecompositions, uses the robust QR factorization for which many efficient parallel (systolic, wavefront array) implementations exist. First, a simple detection scheme is presented and, through simulations, is shown to work very well for sufficient SNR, even when signals are coherent. Outputs of the detection process include simultaneously estimates of signals Direction Of Arrivals (DOA's) and a simple beamformer vector resulting in an estimate of the desired signal.

Extensive simulations are performed assuming different scenarios of variations in SNR, DOA's leading to discussions on the possibilities and limitations of the proposed solution.

Table of contents

PRELIMINARIES AND GENERAL DESCRIPTION OF A PARAMETER DETECTION AND ESTIMATION ALGORITHM

An algorithm with computational advantages is designed for the detection of multiple narrow-band superimposed signals (with sufficient power) in sensor arrays and the simultaneous estimation of some specific signal parameters with a specific digital signal processing scheme on the sensor outputs.

A multidimensional parameter estimation procedure is proposed that can be applied to arbitrary known array geometrical structure and signal correlation. A Gauss-Newton type algorithm is suggested for the minimization of some defined cost function towards the true parameters with fast quadratic convergence if the initialization is close enough.

The parameters could be the azimuth or elevation angle characterizing the Direction Of Arrival (DOA) of the signals, their carrier frequencies, their power, any parameter that relates directly or indirectly to the emitter signals with respect to the array response. In the rest of this report, we will concentrate our presentation principally on the DOA's.

The importance of the knowledge of values of these parameters is sometimes crucial such as in the case of DOA's where some very narrow beams could be formed with the present technology resulting in a substantial capacity increase in some communications systems where the allocated spectrum is being rapidly saturated such as cellular telecommunications or satellite communications. Moreover, this knowledge could be used effectively for a spatial localization especially in cellular telephone systems where hand-overs are often needed.

Applications of the proposed algorithm are in the fields of radio, underwater and seismic array signal processing.

We assume an array of sensors of arbitrary geometry but with known array manifold over the range of parameters of interest; in other words, the response of each sensor to a unit-wavefront signal should be known (through mathematical modeling or experimental measurements) for all possible values of the signal parameter of interest in the specified range.

The number of sensors is assumed greater than the number of present signals (or at least the dominant signals in terms of power) impinging on the sensor array. Moreover, the signals are supposed narrow-band in the sense that the source (envelope) signals remains essentially unchanged in value in the time it takes the plane wave to travel accross the array.

The output signal sensors will depend on the array sensors responses to the signals as well as independent, internally generated noise due principally to thermal noise. Using the spatial filtering or beamforming capabilities of an array of sensors [1], it is possible to imagine steering deep nulls towards the DOA's of signals such that the output of the sensor array will not contain power due to signals but only noise. Since the noise generated in the sensors is usually a white noise (even in the case where it is not, we can always prewhiten our data through the inverse estimate of the noise correlation) and its contribution to the output power does not depend on signal parameters, its value depending rather on the steering vector norm which is constant, the total beamformer output power is always minimal equal to that resulting from noise only. Anytime one or more signals is not nullified (which means that the number of present signals is underestimated), or steered nulls are not done correctly (incorrect DOA value), would result in some contribution of signal power always adding to that of the noise.

This fact is the basic idea which we exploit to detect and estimate the signal parameters leading to the following claims.

Claim 1

Detect the number of signals with relevant power and arbitrary correlation (even coherency) impinging on an array of sensors.

It is known that an array consisting of m sensors has, in general the capacity to generate $(m-1)$ nulls which means that, at least $(m-1)$ signals coming from different DOA's (in theory more, if some signals are coming from identical DOA's) can be simultaneously nullified. In the case when $d<m$ signals are present, it can be deduced that there are at least $(m-d)$ independent sets of m coefficients (steering vector) for which the sum of the products with the sensors outputs nullify the signals at exactly the d DOA's .

This implies that in order to find the correct number and values of DOA's, a multidimensional search over the optimal space generated by these vectors has to be used. By measuring the total output power produced by these beamformers for different possible values of number and values of DOA's, the correct choice is that for which this power is minimal.

The corresponding algorithm is therefore a multidimensional search which can be solved efficiently in terms of convergence speed by using a Gauss-Newton type algorithm. For this type of algorithm, the computation of the first and second derivative with respect to the parameters to estimate of the defined power cost function is necessary. These quantities, respectively the so-called gradient and hessian matrix can be fortunately, well approximated in closed-form formulas. The modified variation projection algorithm combined with a Marquardt-Levenberg step size choice was found to be very successful in converging towards the global minimum when initial estimate were close enough.

Claim 2

Estimate the values of the DOA's of present signals. This is a direct consequence of the previous multidimensional search. The correct values of DOA's are those for which the total power previously described is minimal. This minimum should be the global one, so that special care should be taken in order for the algorithm to converge to the global minimizer. Several DOA initializations at different values of the prescribed range would increase the probability of global convergence.

Claim 3

Estimate the values of power of present signals. This is also a by-product of the calculation of the gradient and the Hessian matrix mentioned above. In the earlier reasoning, steering nulls towards the signals to be able to locate their parameters leads to an ouput power with ideally no signal contribution. The substraction of sensor output before and after null steering should contain more signal information; some additional simple signal processing will be sufficient to separate and identify all present signals. Indeed, it is shown with the help of computer simulations that for sufficient SNR, the performance of this simple method is acceptable and actually, nearly as efficient as a computationally involved estimation operation such as the Linearly Constrained Minimum Variance (LCMV) and behaves better than other simple estimation method like the reference signal method with the additionnal advantage that it can be used in a coherent signal environment. We remind briefly here the beamformer weight for both conventional methods, more details can be further examined in [1]:

for the LCMV method: $\mathbf{w} = \mathbf{R}_x^{-1}\mathbf{C}\left(\mathbf{C}^\dagger\mathbf{R}_x^{-1}\mathbf{C}\right)^{-1}\mathbf{f}$ such that $\mathbf{C}^\dagger\mathbf{w} = \mathbf{f}$

for the reference signal method: $\mathbf{w} = \mathbf{R}_x^{-1}\mathbf{r}_{xd}$

where $\mathbf{R}_x$ is the data correlation, $\mathbf{C}$ is the constraint matrix, $\mathbf{f}$ is the response vector and $\mathbf{r}_{xd}$ is the cross-correlation between the data and the reference signal.

The main advantage of the proposed algorithm is in the efficient computation of the steering vectors by using the so-called *QR* transform of the matrix formed by collecting the array responses to different signal scenarios. Mathematically speaking this transform consists of the decomposition of the array response into the product of a unitary matrix *Q* with a matrix *R* having a special structure in that its last *(m-d)* rows are null whereas its *d* first rows correspond to a triangular matrix, revealing by that the rank of the array response which is equal to the number of signals. The popularity of the *QR* transform is also enhanced by the fact that, not only very fast algorithms (parallel, multiplierless...) could be used for its implementation but also it benefits from its well-known numerical robustness [2]. A multidimensional algorithm is used to converge to the correct *Q* matrix and therefore to the correct DOA's and could be described in the following two major steps of computations:

First step

Assuming some initial minimum number of present signals and initial values of DOA's, we use the Newton-Gaussian type search on the DOA's (further explained in section 3) for which the minimum power is obtained in the last *(m-d)* rows of the corresponding matrix *R* which is our cost function. Next, we increment the number of supposed signals and repeat the same operations as long as the rate of decrease in the value of this minimum is continuing.

The correct number of signals is set as the minimum value for which this rate was decreasing monotonously.

Second step

Actually, during this step, we just collect some quantities already evaluated in the previous one. Among the quantities computed during the minimization process for the approximation of the gradient and the Hessian matrix are the actual values of the parameters to be estimated and the estimates of the signals themselves which are approximated using the new simple beamformind method. This new estimation approach was described in Claim 3 and is further explained in detail in section 3.

The algorithm was implemented in the mathematics software package Matlab for Windows, Version 4.2c1 since routines that perform the required signal processing such as as the *QR* transform, the matrix product, sum, inverse and Schur product are already implemented. Given the matrices dimensions, the dominant matrix operation in terms of time consumption is the *QR* transform which can be implemented much faster than the eigendecomposition of data correlations used in other conventional high resolution detection-estimation algorithms.

The algorithm could also be performed by using a direct hardware implementation that could increase sensibly the speed of execution in a real system especially that there exist parallel numerical architectures for the realization of all required matrix operations.

The organization of the next development is as follows. First, in section 1, the problem at hand is described and mathematical models and assumptions are made for a sensor array. Section 2 follows with the introduction of the new detection scheme which is continued in section 3 by the proposition of the DOA and signal estimation procedure. In section 4, extensive Monte-Carlo simulations are done assuming various possible scenarios, and discussions are developed showing the strengths and limitations of the proposed method. In section 5, we describe the possibility of using several efficient signal processing schemes for an implementation of the proposed method for a trade-off between the accuracy and the speed of the system.

SECTION 1 INTRODUCTION

m sensors receive d ($d < m$) narrow-band plane waves with center frequency ω_0. The sensor signals can be represented as:

$x_i(t) = f(t)\cos(\omega_0(t + \tau_i) + \phi) + n_i(t)$; τ denotes the time required for the plane wave front to travel from the origin to the i^{th} sensor.

The source (envelope) signal $f(t)$ is a real-valued slowly varying function of time, and $n_i(t)$ represents additive noise at the i^{th} sensor. The terminology narrow-band implies that $f(t)$ remains essentially unchanged in value in the time it takes the plane wave to travel accross the array. Using conventional signal processing techniques, the sensor signals are first frequency down-shifted to baseband frequencies. This entails the multiplication of each of the sensor signals by the sinusoids $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ and then low-pass filtering the m product signal pairs. The outputs of each low-pass filter pair are given by respectively the in-phase component $f(t + \tau_i)\cos(\omega_0 \tau_i + \phi_i)$ and the quadrature component $f(t + \tau_i)\sin(\omega_0 \tau_i + \phi_i)$ where $f(t + \tau_i) \approx f(t)$ by using the narrow-band approximation. In the case where d signals are present then the delay τ_i will both have a dependence on the sensor position as well as the direction of the plane wave source i.e. $\tau_{i,k}$ with $i = 1,\ldots,m$ and $k = 1,\ldots,d$. Then, the m-vector sensor outputs may be conveniently represented by the complex-valued array vector: $\mathbf{x}(t) = \mathbf{A}(\tau)\mathbf{s}(t) + \mathbf{n}(t)$ where $\mathbf{s}_k(t) = f_k(t)e^{j\phi_k}$ is the k^{th} signal present and $\mathbf{A}_{pn}(\tau) = e^{j\omega_0\tau_{pn}}$ is the response at the p^{th} sensor of an unit-wave signal with the same plane wave direction as the n^{th} signal.

For the simplicity of the presentation, assuming all directions of travel of plane waves are contained in a plane surface and the sensor array is a linearly uniform array then it can be derived from physical principles that $\tau_{pn} = pl\sin(\theta_n)$ where l is the distance between adjacent sensors and θ_n is the angle between the direction of arrival of the plane wave of interest and the normal to the line formed by the array of sensors at the p^{th} sensor; in what follows we will assume that the vector collecting all DOA's $\theta = (\theta_1 \ \cdots \ \theta_d)$ will be the parameter to be estimated thus we will emphasize the dependance of the array response $\mathbf{A}(\theta)$ directly on θ. On the other hand, $\mathbf{s}(t)$ and $\mathbf{n}(t)$ are taken as respectively the d-vector signal and m-vector white noise with equal power σ^2; the signal and noise vectors are supposed uncorrelated.

Therefore, N measurements snapshots are collected such that:

$$\mathbf{X} = \left(\mathbf{x}(t_1) \quad \cdots \quad \mathbf{x}(t_N)\right) = \mathbf{A}(\theta)\mathbf{S} + \mathbf{N}_0 \tag{1}$$

where $\mathbf{S} = \left(\mathbf{s}(t_1) \quad \cdots \quad \mathbf{s}(t_N)\right)$ and $\mathbf{N}_0 = \left(\mathbf{n}(t_1) \quad \cdots \quad \mathbf{n}(t_N)\right)$

and for large N, the autocorrelation sample matrix $\mathbf{R}_x = \dfrac{1}{N}\mathbf{X}\mathbf{X}^\dagger$ becomes $\mathbf{R}_x \approx \mathbf{A}\mathbf{R}_s\mathbf{A}^\dagger + \sigma^2\mathbf{I}$ where $\mathbf{R}_s$ is the signal autocorrelation.

In our approach, the array manifold is also supposed known through measurements or mathematical modeling of the array response. But, instead of eigendecomposition of the data correlation, we derive a noise subspace via a computationnally efficient QR transform on $\mathbf{A}(\theta) = \mathbf{Q}(d,\theta)\mathbf{R}$. For simplicity, we will drop the θ and (d,θ) factor from $\mathbf{A} = \mathbf{QR}$.

Due to the special structure of $\mathbf{R} = \begin{pmatrix}\mathbf{R}_1 \\ \mathbf{0}\end{pmatrix}$, where $\mathbf{R}_1$ is d by d triangular matrix and $\mathbf{0}$ is the $(m-d)$ by d zero matrix, the following transformation on the received data $\mathbf{Q}^\dagger\mathbf{x}(t) = \mathbf{R}\mathbf{s}(t) + \mathbf{Q}^\dagger\mathbf{n}(t)$ has its last $(m-d)$ components depending on the noise only (the symbol † meaning the transpose conjugate matrix transformation). Supposing that the array is unambiguous i.e. $\mathbf{A}(\theta)$ has full rank for all distinct θ_i of interest, the unitary matrix $\mathbf{Q}$ for which the last $(m-d)$ channels have minimum power will correspond to the correct DOA's.

The Frobenius norm of $\dfrac{1}{\sqrt{N(m-d)}}(\mathbf{0} \quad \mathbf{I})\mathbf{Q}^\dagger\mathbf{X} = \dfrac{1}{\sqrt{N(m-d)}}\mathbf{Q}_2^\dagger\mathbf{X}$ (where $(\mathbf{0} \quad \mathbf{I})$ selects the last $(m-d)$ elements of $\mathbf{Q}^\dagger\mathbf{X}$ therefore $\mathbf{Q} = (\mathbf{Q}_1 \quad \mathbf{Q}_2)$, $\mathbf{Q}_1$ with d columns, $\mathbf{Q}_2$ with $(m-d)$) is set as the cost function $V(\theta,d)$ to be minimized. Observe that the cost function approaches σ^2 the noise power assumed constant at each channel at the global minimum (i.e. at d and θ respectively the correct number of detected signals and the correct DOA's) as the number of data becomes large: $V_{\min}(\theta,d) \approx \sigma^2$.

SECTION 2 DETECTION SCHEME

In this section, we develop the detection algorithm based on the value of the cost function at the minimizer. Suppose the detected number of signals $\hat{d}$ is under-estimated, the minimum of the cost function will forcingly depend not on noise only but also on signals which will also contribute. Indeed, $\mathbf{Q}^{\dagger}\mathbf{R}_x\mathbf{Q} \approx \mathbf{Q}^{\dagger}\mathbf{A}\mathbf{R}_s\mathbf{A}^{\dagger}\mathbf{Q} + \sigma^2\mathbf{I}$ and to simplify the analysis, we will assume uncorrelated signals having comparable power ($\mathbf{R}_s \approx \langle s^2 \rangle \mathbf{I}$ where $\langle s^2 \rangle$ is the average signal power) and values of minimum cost functions occuring for θ belonging to a $\hat{d}$-subset of $\{\theta_1, \theta_2, \ldots, \theta_d\}$. Without loss of generality, we set $\theta = \{\theta_1, \theta_2, \ldots, \theta_{\hat{d}}\}$, therefore:

$$\mathbf{Q}^{\dagger}\mathbf{R}_x\mathbf{Q} \approx \begin{pmatrix}\mathbf{R}' & \mathbf{Q}^{\dagger}\mathbf{a}(\theta_{\hat{d}+1}) & \ldots & \mathbf{Q}^{\dagger}\mathbf{a}(\theta_d)\end{pmatrix}\mathbf{R}_s \begin{pmatrix}\mathbf{R}' & \mathbf{Q}^{\dagger}\mathbf{a}(\theta_{\hat{d}+1}) & \ldots & \mathbf{Q}^{\dagger}\mathbf{a}(\theta_d)\end{pmatrix}^{\dagger} + \sigma^2\mathbf{I}$$

where $\mathbf{R}' = \begin{pmatrix}\mathbf{R}'_1 \\ \mathbf{0}\end{pmatrix}$ with $\mathbf{R}'_1$ triangular $\hat{d}$ by $\hat{d}$ matrix and $\mathbf{0}$ is the $m-\hat{d}$ by $\hat{d}$ zero matrix.

At the global minimum the cost function is approximated by: $\left(\sigma^2 + \frac{1}{m-\hat{d}}\langle s^2 \rangle \sum_{i=\hat{d}+1}^{d} Trace\{\mathbf{a}_i\mathbf{a}_i^{\dagger}\mathbf{Q}_2\mathbf{Q}_2^{\dagger}\}\right)$ where $\mathbf{a}_i = \mathbf{a}(\theta_i)$. Obviously, the minimum σ^2 is attained when the sum in the previous equation is null i.e. when $\hat{d} = d$. This works especially well for high SNR; for lower SNR, the second term in the approximation will have a lesser contribution therefore a differential cost function between successive detection measurements will be less affected by the σ^2 factor and is shown empirically to work better. Instead of measuring the minimum of the cost function for different values of $\hat{d}$, we compare its slope with respect to increasing value of $\hat{d}$, and the detected number of signals corresponds to the value for which this slope decreases as compared with the previous one. In summary, the proposed algorithm could be viewed as a minimization over the number of signals calling another minimization over the signal parameters to estimate as described by the following relation.

$$\min_d\left\{\min_\theta\{V(d,\theta)\}\right\} = \min_d\left\{\min_\theta\left\{\frac{1}{N(m-d)}\left\|\begin{pmatrix}\mathbf{0} & \mathbf{I}\end{pmatrix}\mathbf{Q}^{\dagger}\mathbf{X}\right\|_F^2\right\}\right\}$$

The first minimization is a brute force uni-dimensional search over increasing discrete values of d; for each value of d, next a sub-routine is called and performs the multi-dimensional search over the space of signal parameters (in this case, the DOA's θ) that permits to converge to some minimal cost function value (because of the multidimensional aspect, careful initializations have to be done). This value is compared to the two previous ones to find the maximum decreasing slope for which the decision of correct detection is made. Initializations for both minimizations are necessary; notes 1 and 2 below give some general guidelines for the choice of initial values respectively for the first and the second minimization procedures.

The detection algorithm is outlined below.

Summary of the algorithm:

1) Initialize $\hat{d}$ to some value; also, initialize or compute the minimal value of the cost function $V(d,\theta) = \frac{1}{N(m-d)} \left\| (\mathbf{0} \quad \mathbf{I})\mathbf{Q}^{\dagger}\mathbf{X} \right\|_F^2$ for $d = \hat{d} - 1$ and $d = \hat{d} - 2$ (more details about how to make the initialization choices are developed in note 1).

2) Minimize over $\theta = \left\{\theta_1, \theta_2, \ldots, \theta_{\hat{d}}\right\}$ the cost function

$$V_{\min}(\hat{d}, \theta) = \min_{\theta} V(\hat{d}, \theta) = \min_{\theta} \frac{1}{N(m-\hat{d})} \left\| (\mathbf{0} \quad \mathbf{I})\mathbf{Q}^{\dagger}\mathbf{X} \right\|_F^2$$

3) Null Hypothesis H_0: $d = \hat{d}$

4) If $V_{\min}(d, \theta) - 2V_{\min}(d-1, \theta) < -V_{\min}(d-2, \theta)$ then

reject H_0, let $\hat{d} = \hat{d} + 1$ and go to 2

else accept H_0 stop

Note 1

For applications of repetitive nature where reception of signals is continuous, the initial value for $\hat{d}$ could be set as the minimal number of present signals expected at any time. For example, in mobile communications radio systems, each mobile in a cluster is assigned one particular frequency so that interferences can come only from mobiles of neighboring clusters using the same frequency; since in usual cluster architectures, six (6) adjacent interfering clusters exist, we could initialize $\hat{d}=7$ (1 desired signal plus 6 interferers). The actual correct number usually is larger due to additional non-adjacent interfering clusters plus multipath effects. In the case where no a-priori information about the number of present signals exist, a brute force computation of initial cost function values for $\hat{d}=1$, $\hat{d}=2$ and $\hat{d}=3$ would be necessary to start running the proposed algorithm.

Note 2

Because of the multi-dimensional nature of the parameter minimization procedure, initial estimates have to be close to actual ones (in the simulations presented in Section 4, it was found that initial values of DOA's different from the actual ones by at most 10 degrees lead to global convergence). Some guesses could be made for these initial values for some applications where the receiver knows approximately the directions of interferences and desired signal. For example, in mobile radio communications systems, due to geometrical considerations, positions of interfering cluster cells with respect to the base-station imposes some fixed interval for possible variations of interfering DOA's. In the case where no guesses can be made, then it is possible also to run several times the parameter minimization procedure for different initial values and choose the outputs with the smallest minimum cost function. Another advantage that we exploit is that since the search procedure is repeated for increasing values of d, we can use the output estimates of the previous minimization process as initial values for the present one which, because of the increase of the problem dimension requires an additional initial value that can be set according to the previous scheme. This is an application of the alternating projection algorithm discussed in more details in [3].

It is noteworthy to point out that, at the end of the detection scheme, we have available not only the number of present signals but also the DOA's, the noise power and values of matrices $\mathbf{Q}$ and $\mathbf{R}$, hopefully at the correct detected number of signals and at the correct DOA's, that will be used in the signal estimation process. In section 4, a simulation example is provided testing the performance of this detection scheme.

SECTION 3 DOA AND SIGNAL ESTIMATION

In the previous detection scheme, although no eigendecomposition is performed, a multidimensional search procedure is needed to look for the global minimizer of the cost function. The modified variable projection algorithm, which converges quadratically in $O(md^2)$ multiplies [4] when initialized closely enough to the global minimizer, is used. Similarly as in [4], the following operations are needed during the optimization process.

- *QR* decomposition

$$\mathbf{A} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix} \tag{2}$$

- Intermediate variables that are needed in the course of computing the gradient and the hessian matrix during the optimization process

$$\mathbf{D} = \begin{pmatrix} \frac{\partial \mathbf{a}(\theta_1)}{\partial \theta_1} & \cdots & \frac{\partial \mathbf{a}(\theta_d)}{\partial \theta_d} \end{pmatrix}$$

(Here we point out that the computation of the *QR* factorization as well as $\mathbf{D}$ could be available ahead of time since $\mathbf{A}$ is required to be known a-priori through measurements or mathematical modeling)

$$\Phi = \mathbf{Q}_2^{\dagger}\mathbf{D} \tag{3}$$

$$\Psi = \mathbf{M}^{\dagger}\mathbf{Q}_2 \tag{4}$$

$$\Omega = \mathbf{R}_1^{-1}\mathbf{Q}_1^{\dagger}\mathbf{M} \tag{5}$$

where $\mathbf{M} = \frac{1}{\sqrt{N(m-d)}}\mathbf{X}$.

The expressions in equations (4) and (5) can be interpreted physically as respectively the error funtion whose power has to be minimale at the correct parameter values and the outputs of estimated signals as will be shown later in this section.

- Criterion function V, gradient $\mathbf{V}'$, Hessian matrix $\mathbf{H}$ and search direction $\mathbf{s}$ correspond after some mathematical manipulations to:

$$V = Trace\left\{\Psi\Psi^{\dagger}\right\} \tag{6}$$

$$\mathbf{V}' = 2\,\mathrm{Re}\left\{diag(\Omega\Psi\Phi)\right\} \tag{7}$$

$$\mathbf{H} = 2\mathrm{Re}\left\{\left(\Phi^{\dagger}\Phi\right)\otimes\left(\Omega\Omega^{\dagger}\right)^{T}\right\} \tag{8a}$$

$$\mathbf{H}_{ii} = \mathbf{H}_{ii}(1+\mu_k) \tag{8b}$$

$$\mathbf{s} = \mathbf{H}^{-1}\mathbf{V}' \tag{9}$$

the estimate is iteratively computed as:

$$\theta_{k+1} = \theta_k - \mathbf{s} \tag{10}$$

where the symbol $\otimes$ represents the Schur product; moreover, μ_k was chosen according to the Levenberg-Marquardt step length technique [2]. It consists of choosing the particular step size in such a way that the minimization process behaves as the steepest descent method when the method is far from the minimum and as an inverse-Hessian method when the minimum is approached. Practically, we increase μ_k by a factor of 10 when the cost function increases and decrease it by a factor of 10 when the cost function decreases (see [2, Chapter 15] for more details).

The initialization process can be enhanced using the alternating projection [3] algorithm which is an application of the relaxed optimization principle ("one parameter at the time") to the optimization problem [4]. In our case, it can be naturally applied since the search procedure is repeated for increasing values of d so that we can use the output estimates of the previous minimization process as initial values for the present one which, because of the increase of the problem dimension requires an additional initial value chosen according to Note 2 in section 2. In any case, the optimization is the bottle-neck problem in terms of computation cost since the eigendecomposition is not used. The resulting algorithm is described in the flow graph of Fig. 1.

In the next section, different possible scenarios of locations of sources and SNR are examined through computer simulations using the mathematics software Matlab. Results give satisfactory detection and estimation outputs as long as the SNR is greater than some threshold and the signal's separation is enough. For cases where these conditions are not satisfied and especially when two DOA's become very close to each other, a more accurate initialization procedure is necessary. In a mobile communications system context, since locations of sources do not move too far between detection operations, a better initialization process can be made by taking previous DOA's estimations as initial estimates of next ones. When DOA's are well separated, our algorithm produces estimates that are precise enough to be fed as initial values for the next estimation procedures and a better convergence is obtained in terms of DOA resolution.

Next, since the basic contribution of this work is to simplify calculations without degrading too much performances, a simple beamforming method is proposed and tested. From equations (1) and (2), $\mathbf{R}_1^{-1}\mathbf{Q}_1\mathbf{X} = \mathbf{S} + \mathbf{R}_1^{-1}\mathbf{Q}_1\mathbf{N}$ where $\mathbf{S}$ is the signals snapshots, an approximation is proposed such as $\hat{\mathbf{S}} = \mathbf{R}_1^{-1}\mathbf{Q}_1\mathbf{X}$. Another advantage in this estimate is that it is already available from equation (5) in the detection process so that no additionnal calculation is required.

It is shown in the simulation section that when SNR is high enough and when DOA's are not too close, our estimation procedure is as performant as such computation-involved estimation procedures such as the Linear Constrained Minimum Variance (LCMV) method or the reference signal method [1]. To compare different beamforming performances, the input SNR and SINR (signal to noise plus interference ratio) are measured at the first sensor with response $a_1(\theta)$ whereas the output SNR and SINR are measured using the proposed beamforming estimators.

The input signal to noise plus interference ratio (ISINR) and the output signal to noise plus interference ratio (OSINR) are defined respectively as:

$$ISINR = \frac{E\left\{\left|a_1(\theta_1)s_1(t)\right|^2\right\}}{\sigma^2 + \sum_{k=2}^{\hat{d}} E\left\{\left|a_1(\theta_k)s_k(t)\right|^2\right\}} \quad \text{and}$$

$$OSINR = \frac{E\left\{\left|\mathbf{w}^*\mathbf{a}(\theta_1)s_1(t)\right|^2\right\}}{\sigma^2\left|\mathbf{w}\right|^2 + \sum_{k=2}^{4} E\left\{\left|\mathbf{w}^*\mathbf{a}(\theta_k)s_k(t)\right|^2\right\}}$$

$E\{.\}$ denotes the statistical expected value and $\mathbf{w}^*$ the corresponding steering vector.

The difference between these two quantities represent the improvement in the signal estimation procedures by using adaptive beamforming.

SECTION 4 SIMULATION

In this section, some simulation examples are provided to test the accuracy of the present method estimates as compared with WSF ones under different scenarios.
An 8-element half-wavelength spaced linear uniform array is assumed and 4 signals with equal power are located at different DOA's. Each signal and noise vector is independently generated in the computer using the Matlab random Gaussian generators initialized with different seeds. In each experience, the data vector dimension is 100 corresponding to the supposed number of snapshots and each point in the figures resulting from averaging 200 signals and noise realizations. For the case of coherent signals, corresponding data vectors are initialized with the same seed value in the random generator thus assuming identical signals.

Example 1

In this case, the detection scheme using the proposed approach is tested assuming stationary DOA's at -30°, 20°, 30° and 60°. The cost function evaluated in dB at the minimizer DOA's is plotted against increasing values of d for SNR between -10 dB to 20 dB in Fig. 2. As expected, for each SNR > 8 dB, the minimum cost function is obtained for the correct number of signals $d = 4$, and detecting this minimum could be sufficient for such SNR's. But, by using the scheme described in section 2 where values of the slope of the cost function versus successive detected number, even under lower SNR > -6 dB, the detection is possible. The algorithm has worked equally well for both uncorrelated and completely correlated (coherent) signals. For the coherent case, the signals at 20° and 30° are assumed exactly the same.

Example 2

In this case, assuming that the correct number of signals is detected, with the same stationary experiment as before where DOA's are at -30°, 20°, 30° and 60°, the performance of our DOA estimation procedures is plotted in Fig. 2. In Fig. 3(a), the standard deviation error of the signal coming at DOA 30° is shown against the variation of SNR for the incoherent case, and the present method is compared to the WSF method. In the range of SNR shown and for the given DOA's separation, the WSF and proposed approaches have comparable performances. In Fig. 3(b), we observe similar performances for the coherent case.

Example 3

In the next experiment, for both our method and the WSF one, one source location was varied between 20° and 26° (the others kept constant) and we repeated measurements of the detection failure percentage for each value of SNR between -5 dB and 10 dB. Resulting data were too lengthy to include in this paper and instead comments on their significance with respect to variations in SNR values or DOA positions follow. In particular, the detection failure is less than 15 % for a difference between DOA's greater than 5°; when this is less than 4°, the failure can be as large as 80 % for the proposed approach. For comparison, the WSF detection failure does not exceed 50 % for a DOA separation greater than 2° and an SNR > 3 dB, and is always less than 20 % for a DOA separation greater than 4°.

It was found experimentally that the main reason for the detection failure for close DOA's was poor initial estimates. Since in real applications, the estimation process is continuous and DOA's do not vary too much between estimations, the following initialization was used. Results of DOA's outputs of the previous estimation process become the initial estimates of the present one. In the computer simulation, the DOA location to estimate is supposed to move 1° at each estimation procedure starting from the initial position of 20° with the 3 other DOA's fixed at -30°, 30° and 60°. In this case, the results of the simulations, consisting also, for each scenario, of averages of 200 experiments, have proved a detection failure of less than 10% for a DOA difference greater than 4° and an SNR between -5 dB and 10 dB.

Example 4

Now, the signal estimation method presented in section 3 is tested under different SNR and DOA's constraints and compared to the LCMV and the reference signal method approaches. First, similarly as before, we assume SNR=0 dB, 3 fixed DOA's at at -30°, 30° and 60° and the last one moving from 20° to 26°. The ISNIR, under these conditions is approximately constant close to -6 dB, the resulting OSNIR is plotted in Fig. 4(a), where the moving signal is estimated versus various DOA's locations for the coherent (in this case the same signal is also coming from at 30°) and incoherent cases. It was observed that the estimation performance is practically the same for the coherent and incoherent cases for the proposed and the LCMV methods; of course, the reference signal method produced useless estimates under coherent signals so that only results from the non-conherent case were plotted. The gain that can be attained by using the adaptive array antenna for this case varies between 14 and 7 dB and for the sake of comparison, in the same experiment, plots of the reference signal method (the reference signal is supposed perfect in the simulations) and the LCMV method are also included. Results show comparable performances in the range of SNR at hand; for lower SNR's, the performance degrades appreciably in the proposed method, and this is expected since the DOA's estimation becomes inaccurate in this case. The advantage in using DOA's detection based algorithms as compared to the reference signal method is, in addition to the fact that the reference signal performance is very poor under coherent signals, the knowledge of DOA's which can be very useful in a cellular mobile communications systems especially when hand-overs occur. The proposed method could be seen as a trade-off between the LCMV method which also provides for the DOA's values but require expensive computations and the reference signal method which can be computationnally attractive but lacks DOA information.

In Fig. 4(b), the same experiment is repeated for an SNR of 10 dB where ISNIR=-5 dB and of course, the performance is better, a maximum and a minimum gain of respectively 22 dB and 16 dB are possible for a DOA difference of 10 and 4° respectively.

SECTION 5 SIGNAL PROCESSING DESCRIPTION

In our development of signal processing architectures, we have attempted to keep in mind two main constraints for digital real-time applications. First, the speed of the operation execution resulting from parallel, pipelinable structures and second, the accuracy of the numerical results by using robust well-conditioned algorithms. The degree to which a system of linear equations is ill-conditioned is determined by the condition number of the coefficient matrix. The condition number of a matrix $\mathbf{Y}$, $Cn(\mathbf{Y})$ is the ratio of its largest to its smallest singular value; the larger $Cn(\mathbf{Y})$ is, the more ill-conditioned will be the problem to solve.

The signal processing units (SPU) that we need in our algorithm can be divided in the following parts.

Part 1: SPU to get the data ready for digital signal processing after reception by the array. We have opted to use directly the data matrix instead of computing the autocorrelation $\mathbf{R_x}$ [5]. In fact, it is numerically advantageous since $Cn(\mathbf{X}) = \sqrt{Cn(\mathbf{R_x})}$.

Part 2: SPU to QR decompose $\mathbf{A}$ and get the values for $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{R}_1$ and $\mathbf{R}_1^{-1}$for different values of initial DOA's and number of detected signals.

Two possible algorithms for efficiently implementing such a transform are the so-called Given's and Householder's which we describe here briefly since they are well documented [5] and [6].

The Given's rotation applied to the left (respectively the right) of a matrix combines 2 rows (respectively 2 columns) to zero one component in the following form.

$$\begin{pmatrix} c & s^* \\ -s & c^* \end{pmatrix}\begin{pmatrix} 0 & \cdots & 0 & x_i & \cdots & x_k \\ 0 & \cdots & 0 & y_i & \cdots & y_k \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & x'_i & \cdots & x'_k \\ 0 & \cdots & 0 & 0 & \cdots & y'_k \end{pmatrix}$$

where the rotation coefficients c and s satisfy

$$-sx_i + cy_i = 0$$
$$s^*s + c^*c = 1$$
$$c^* = c$$

A sequence of such elimination operations may be used to triangularize a matrix $\mathbf{Y}$ (with dimensions n_1 by n_2). If $n_1 > n_2$, then they are performed to the left of $\mathbf{Y}$ and successive multiplications create zeros on columns of $\mathbf{Y}$ from bottom to top, and from left to right.

Such a transform is applied to the *QR* decomposition of **A** and $\frac{2m-d-1}{2}d$ planar rotations are needed to obtain the form of equation (2). It is noteworthy to mention the existence of multiplierless efficient algorithms to perform rotations [7]; moreover, the Given's algorithm can be naturally applied in a parallel pipelinable architecture such as those described in [5],[6] for an increase of throuput.

The second algorithm that can be used to implement the *QR* decomposition is the Householder's. The transform is a unitary, symmetric matrix $\mathbf{P} = \mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^{\dagger}}{\mathbf{u}^{\dagger}\mathbf{u}}$ such that $\mathbf{P}\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1$ with $\mathbf{u} = \mathbf{x} \pm \|\mathbf{x}\|\mathbf{e}_1$. Therefore, it may act on some column to simulaneously zero all elements below some row until the whole triangularization is complete. For a parallel and pipelinable structure, the Given's algorithm is preferable since it leads to simpler processing units as compared with the Householder transform.

Part 3: The basic linear operations needed in the optimization procedure are expressed in equations (2) through (10). We have 8 matrix multiplications, the most computationally involved of them takes an order of $O(md^2)$ complex operations. This may be compared to the *QR* decomposition that consumes also the same order of calculations. We have also to perform 2 matrix inversions, one of them a triangular matrix inversion with $O(d^3)$ operations and a Shur-product with d^2 operations.

Since it is always assumed that $m > d$ then the cost of signal processing in terms of complex operations is dominated by the $O(md^2)$ term for the proposed method which can be compared with $O(m^3)$ operations for a usual eigendecomposition. For the sake of illustration, Fig. 5(a) and (b) show the gain in computatinal savings by using the *QR* decomposition (md^2 operations) instead of the eigendecomposition (m^3operations) assuming first that $d = 7$ then $d = m-2$, in both cases m is varying between 8 and 20; the gain is especially significant when d is small as compared with m as shown in Fig. 5(a).

This cost could be decreased by using parallel architectures such as the wave-front array of [8] for the computation of the matrix multiplication or matrix inversion as well as the *QR* decomposition. Another possible choice would be the systolic array of [5] for the triangular matrix inversion as well as the *QR* decomposition.

CONCLUSION

A method for the detection-estimation problem in antenna arrays, having high resolution capabilities for SNR > -5 dB, was presented. The method which works also for coherent signals does not require eigendecompositions which leads to higher execution processing and is easily implemented using parallel processing techniques. Instead of an eigendecomposition of the data autocorrelation taking an order of $O(m^3)$ complex operations, a proposed efficient complex Householder or Given's decomposition of $\mathbf{A}(\theta)$ takes $O(md^2)$ operations and even less when parallel processing such as systolic and wave front arrays are used. A multidimensional search procedure is required to find he minimum power for the selected channels. In the special context of cellular mobile communications, an initialization of DOA's based on previous estimates in the optimization process resulted in more accuracy and resolution in the detection method.

A simple signal estimation method whose processing is done during the detection procedure is shown to lead to accurate estimates in some SNR range interval and DOA's separation in space as compared to some expensive estimation methods such as the LCMV.

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LIST OF CAPTIONS

```
%Software program written in Matlab for Windows version 4.2c.1
%Detection of the number of signals impinging in a ULA half-wave
%spaced and estimation of the Direction Of Arrival for each signal
%using the proposed method based on a QR factorization of the array
%manifold.
clear
num_sig_real=('Input the actual number of signals');
%Input the locations of the DOA's in ascending order
teta=[-30 20 30 60]*pi/180;
SNR=input('Signal to Noise Ratio');
gain=10^(-SNR/20);
m=input('Number of sensors');
N=input('Number of snapshots');
%Initialize DOA's
init=[-20 15 40 70 0 -50 -70 50]*pi/180;
fprintf('Minimum Guessed Number of Signals less than %2d',num_sig_real)
num_sig_min=input('');
det_choi=menu('Choose a Detection Method','Differential Detection','Minimal Detection');
num_exp=1;                  %Set the number of experiments
%Initialize the uniform and normal random number generators
randn('seed',273841);rand('seed',86977);
%Function computing the array manifold given m, d and DOA's
a=arr_man(teta,m,num_sig_real,0);
sig_coun=0;                 %Initialize signal counter
pro_end=1;                  %Flag to stop the program when correct decision is made
d=num_sig_min;
while pro_end==1
        sig_coun=sig_coun+1;
        for lex=1:num_exp
%Randomly generate the signals and noise
                s=randn(num_sig_real,N)*exp(j*(2*rand(N)-ones(N))*pi);
                nn=gain*randn(m,N)*exp(j*(2*rand(N)-ones(N))*pi);
%Generate the data according to the ULA model
                x=(a*s+nn)/sqrt(N);
%Initialize cost function
                cos_fun=2e30;   %Cost funtion initialization
%Flag to stop the multidimensional search after convergence or failure
                dfail=1;
%Function for the multidimensional search over the DOA space
%using the QR transform of the array manifold a
                DOA_sear
                if d>num_sig_real
                        dummy=num_sig_real;
                else
                        dummy=d;
                end
                res_DOA(sig_coun,1:d)=ang1;
                res(lex)=cos_fun;
        end
%Routine to check if the correct number of signals has been
%detected using two approaches: the differential and the minimal
%method. The first one more accurate consists in detecting the
%largest decrease of slope in the value of the cost function, in
%the second one, the minimum value of the cost function is detected.
        DOA_det
        clear res;
        d=d+1;
end
```

```
%Display results
disp('The number of detected signals is')
sig_num
disp('The mean values of DOA"s are (in degrees)')
res_DOA_fin(sig_coun-1,1:sig_num)*180/pi
if input('Do you want to beamform ?')=='y'
        beam_form
end
```

```
%Function routine arr_man.m to compute the array manifold of a uniform linear array with %m sensors and half-wavelength spacing
function am=arr_man(angl,m,dsig,par)
if par==0
%Compute array manifold
	for ll=1:m
		for lll=1:dsig
			am(ll,lll)=exp(-j*(ll-1)*pi*sin(angl(lll)));
		end
	end
else
%Compute array manifold derivative with respect to DOA's
	for kk=1:m
		for kkk=1:dsig
			am(kk,kkk)=-j*(kk-1)*pi*cos(angl(kkk))*exp(-j*(kk-1)*pi*sin(angl(kkk)));
		end
	end
end
```

```
%Routine DOA_sear.m to perform the multidimensional search of the DOA's %using a Levenberg-Marquardt
method
count=0;
%DOA and increment initializations
ang=init(1:d);
ss=ones(d,1);
lamda=.001;
while count<20 & max(abs(ss))>1e-4*pi/180 & dfail==1
        count=count+1;                  %Number of loops
        aa=arr_man(ang(1:d),m,d,0);         %Intermediate array manifold
%QR factorization on the array manifold
        [q,r]=qr(aa);
        q2=q(:,d+1:m);
        q1=q(:,1:d);
        if cond(r(1:d,:))>1e10
                dfail=0;
        else
%Intermediate variables calculation
                r1=inv(r(1:d,:));
                dd=arr_man(ang(1:d),m,d,1);
                phi=q2'*dd;
                gamma=r1*q1'*x;
                hessk=2*real((phi'*phi).*(gamma*gamma').');
        end
        if cond(hessk)>1e6 | dfail==0
                count=40;
                dfail=0;
        else
                hessk=hessk+lamda*diag((diag(hessk)));
                pa=x'*q2;
                vpri=2*real(diag(gamma*pa*phi));
                ss=inv(hessk)*vpri;         %DOA increment
                if max(abs(ss))<1e-3
                        dfail=1;
                end
%Update of the cost function
                nornew=real(trace(pa'*pa));
                if nornew<cos_fun
                        cos_fun=nornew;
                        lamda=lamda/10;
%Update of the DOA values
                        ang(1:d)=ang(1:d)+ss';
                else
                        lamda=10*lamda;
                end
        end
end
%For given value of number of signals, values of estimated DOA's
ang1=sort(ang(1:d));
```

```
%Routine DOA_det.m to collect relevant quantities to start the detection process.
%Measure the cost function at the minimizer for increasing number of signals
res_cos_fun(sig_coun)=10*log10(mean(res)/(m-d));
%Saves values of the cost function and DOA's at the minimizers
if sig_coun>1
        res_DOA_fin(sig_coun,1:d)=mean(res_DOA);
else
        if sig_coun==1
                res_DOA_fin(sig_coun,1:d)=res_DOA;
        end
end
if det_choi==1
%Differential method (test the cost function slope)
        if sig_coun>2
                if res_cos_fun(sig_coun)-2*res_cos_fun(sig_coun-1)>-res_cos_fun(sig_coun-2)
                        sig_num=sig_coun-1;
                        pro_end=0;      %Stops the program
                end
        end
else
%Minimal method (test the cost function minimum)
        if sig_coun>1
                if res_cos_fun(sig_coun)>res_cos_fun(sig_coun-1)
                        sig_num=sig_coun-1;
                        pro_end=0;      %Stops the program
                end
        end
end
```

%Routine beam_form.m to simulate the proposed beamforming method compared %to a reference signal and the LCMV methods.

%Compute the array manifold according to the estimated DOA's.

```
aa=arr_man(res_DOA_fin(sig_coun-1,1:sig_num),m,d,0);
k=input('Position of desired DOA angle');
iden=eye(k);
```

%Find the beamformer elements for the proposed approach w1, *the reference signal method* w2 *and the LCMV method* w3

```
[q,r]=qr(aa);
dummy=inv(r(1:sig_num,:))*q(:,1:sig_num)';
w1=dummy(k,:)
irx=inv(x*x'/N);                %Inverse sample covariance matrix
rd=x*s(k,:)'/N;                 %Sample cross-correlation matrix
w2=irx*rd;
f=iden(:,k);
w3=irx*aa*inv(aa'*irx*aa)*f;
w=input('Which beamforming method w1 (proposed), w2 (signal reference) or w3 (LCMV)');
```

%Compute the Output and the Input Signal to Interference plus Noise Ratio

```
for inc=1:sig_num
        sig_aft_beam(inc)=abs(w'*aa(:,inc)*s(inc,:)).^2;
        sig_bef_beam(inc)=abs(aa(:,inc)*s(inc,:)).^2;
end
```

%Compute the values of the output and input SINR

```
osinr=sig_aft_beam(k)/(sum(sig_aft_beam(1:k-1))+sum(sig_aft_beam(k+1:))+ g^2*w'*w);
isinr=sig_bef_beam(k)/(sum(sig_bef_beam(1:k
```

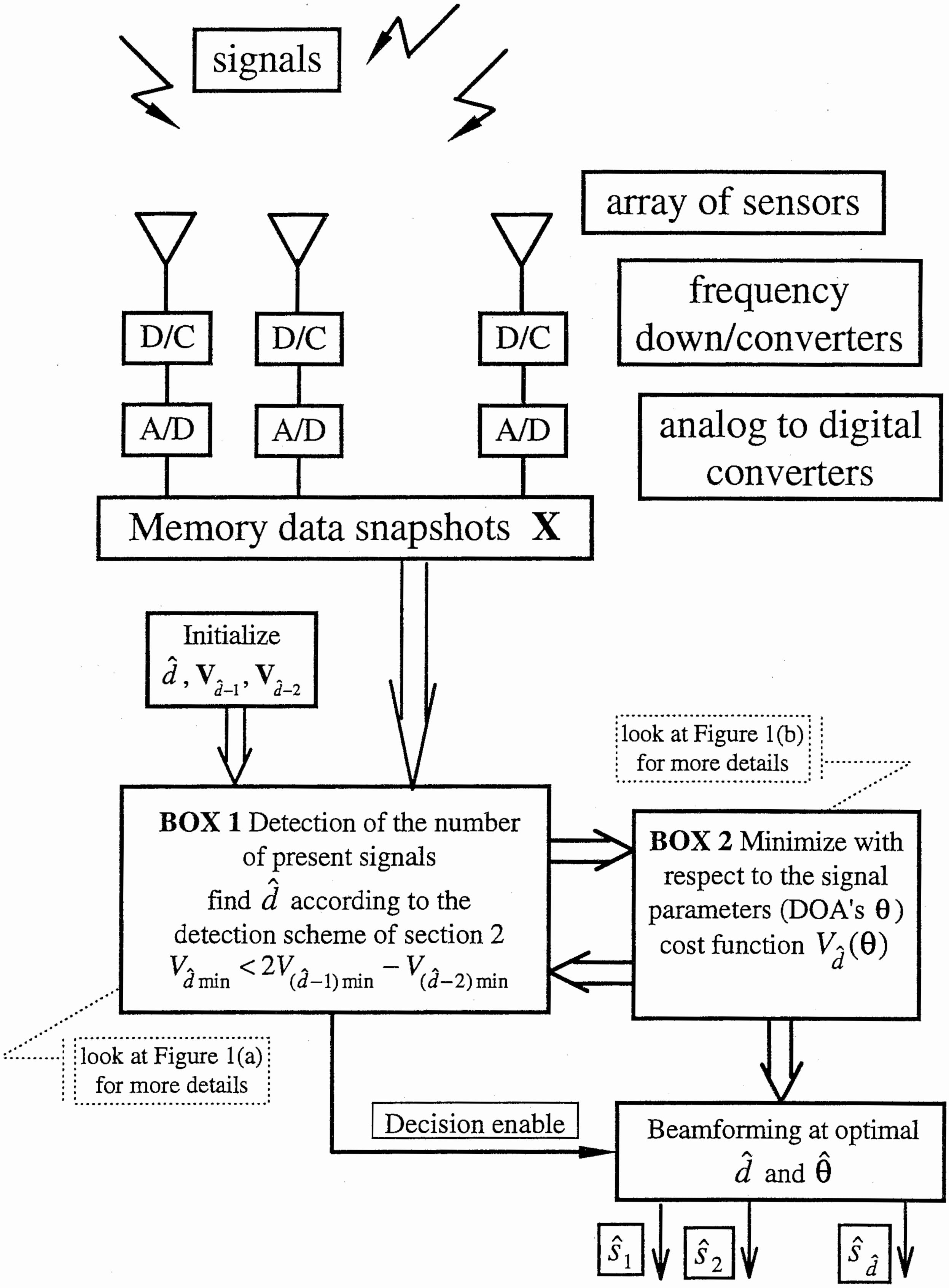


Figure 1 **Flow Graph diagram of the proposed QR based detection-estimation algorithm.**

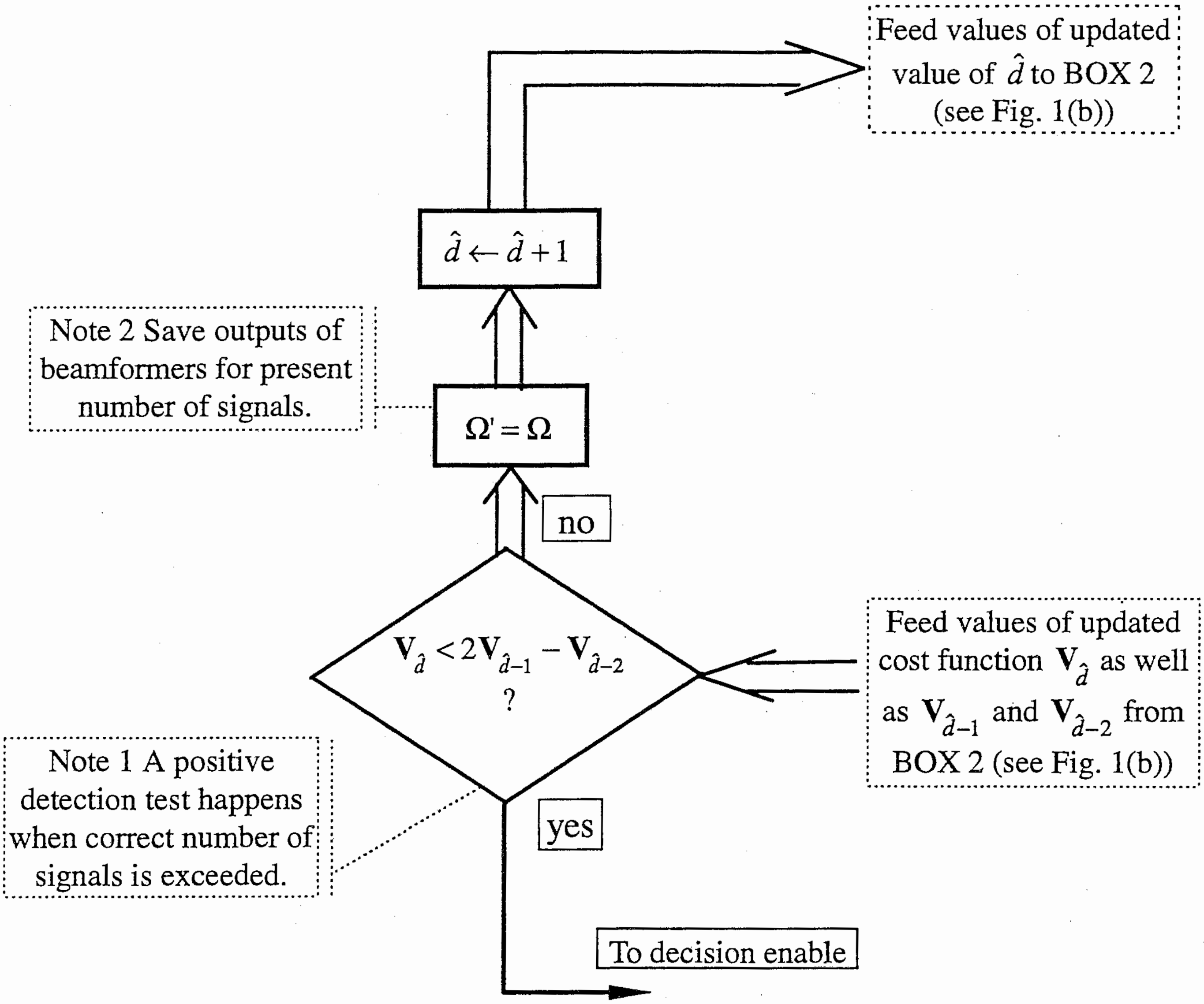


Figure 1(a) BOX 1: Flow Graph diagram of the detection of the number of present signals.

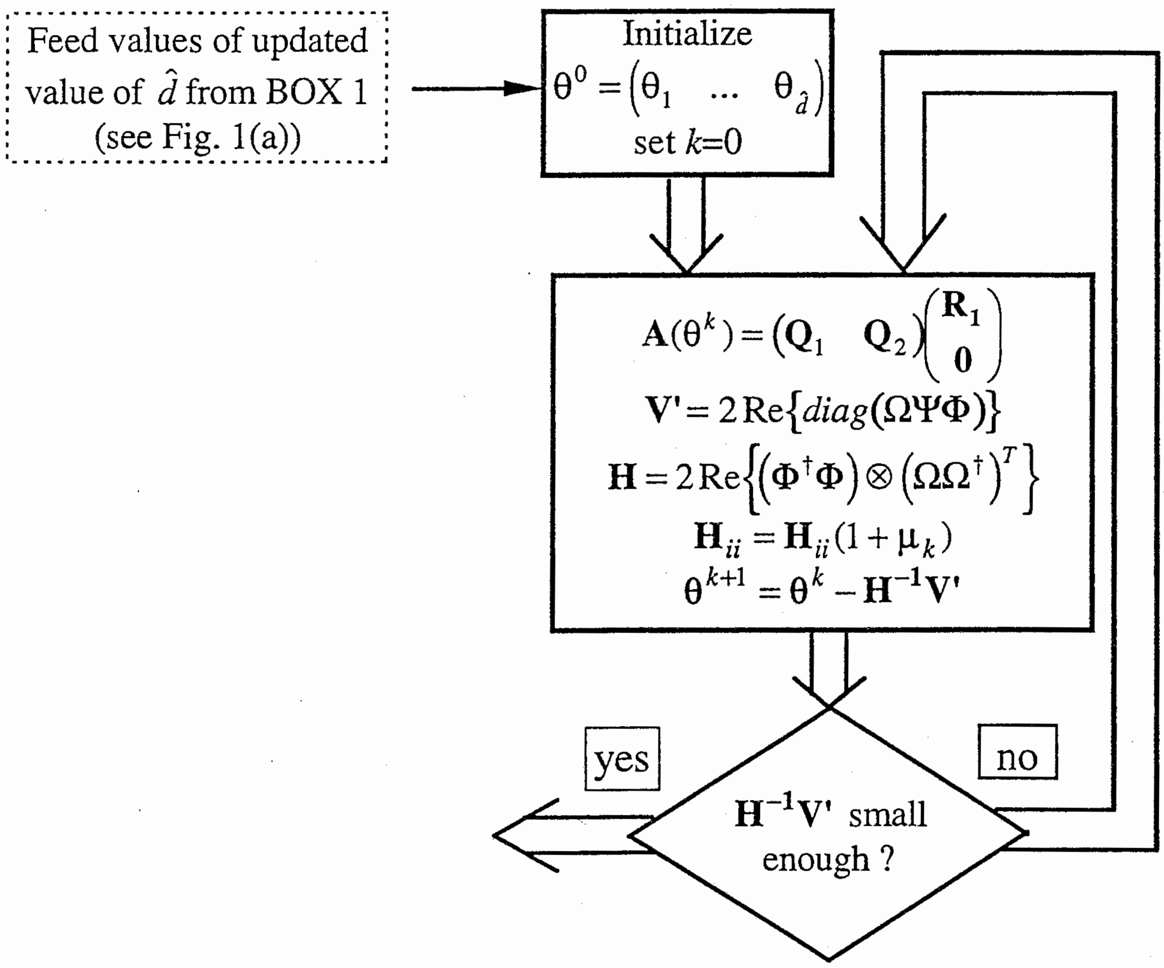


Figure 1(b) BOX 2: Gauss-Newton type search algorithm where V' and H are the approximate gradient and the hessian matrix.

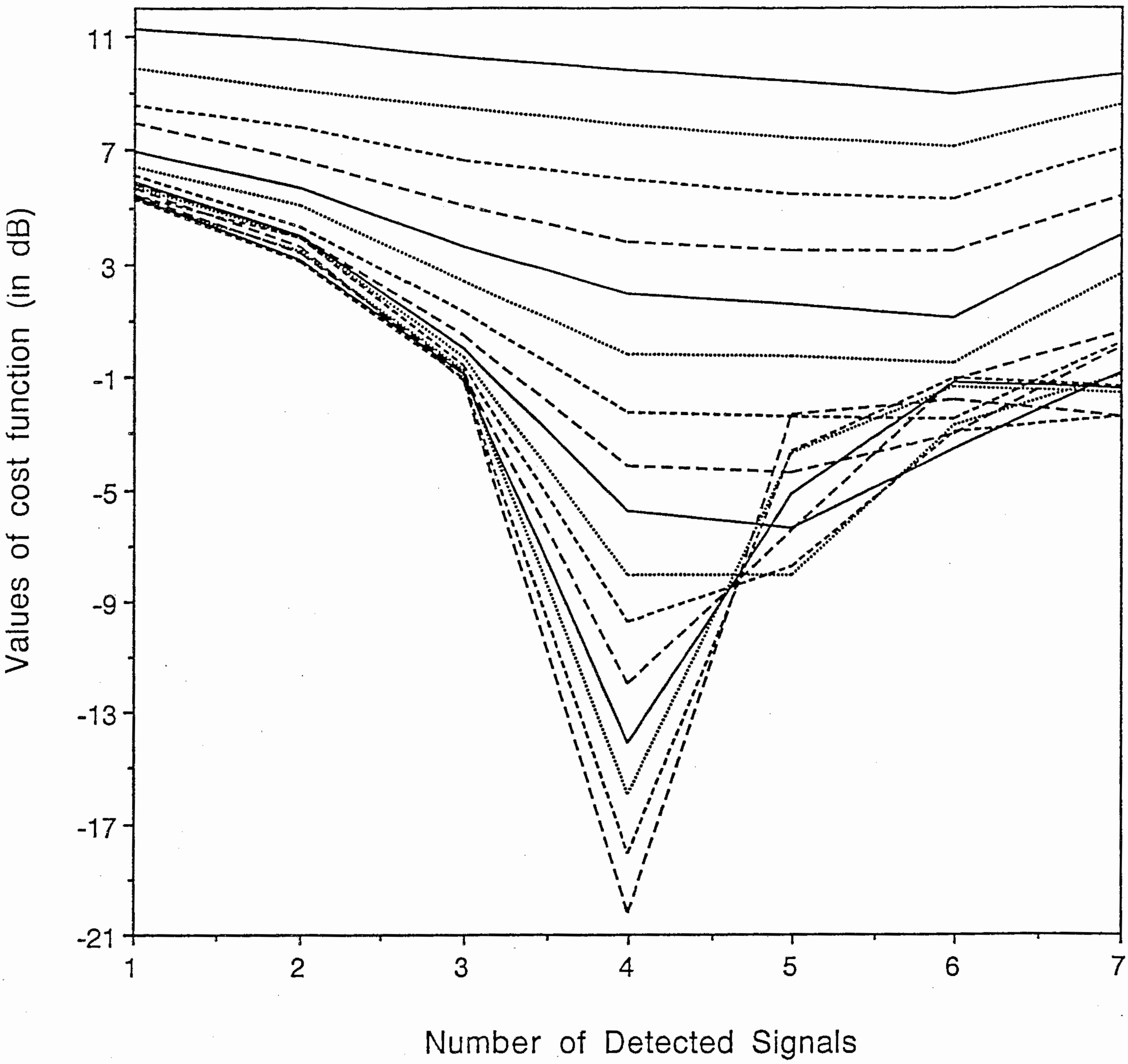


Figure 2 Plots of inverse cost function versus number of detected signals for values of SNR ranging from -10 to 20 dB in steps of 2 dB

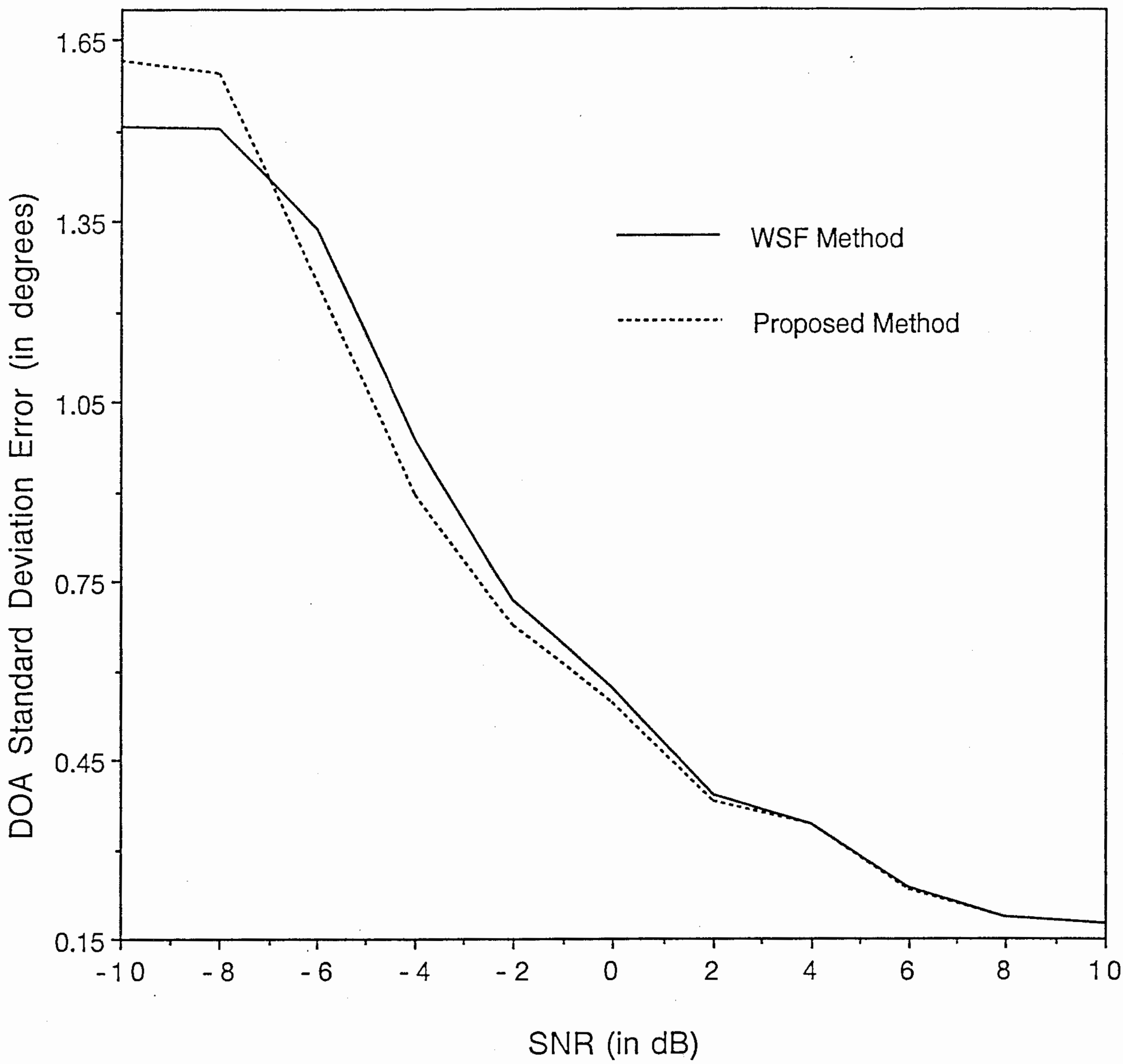


Figure 3(a) DOA Standard Deviation Error versus SNR for both WSF and proposed methods for incoherent signals

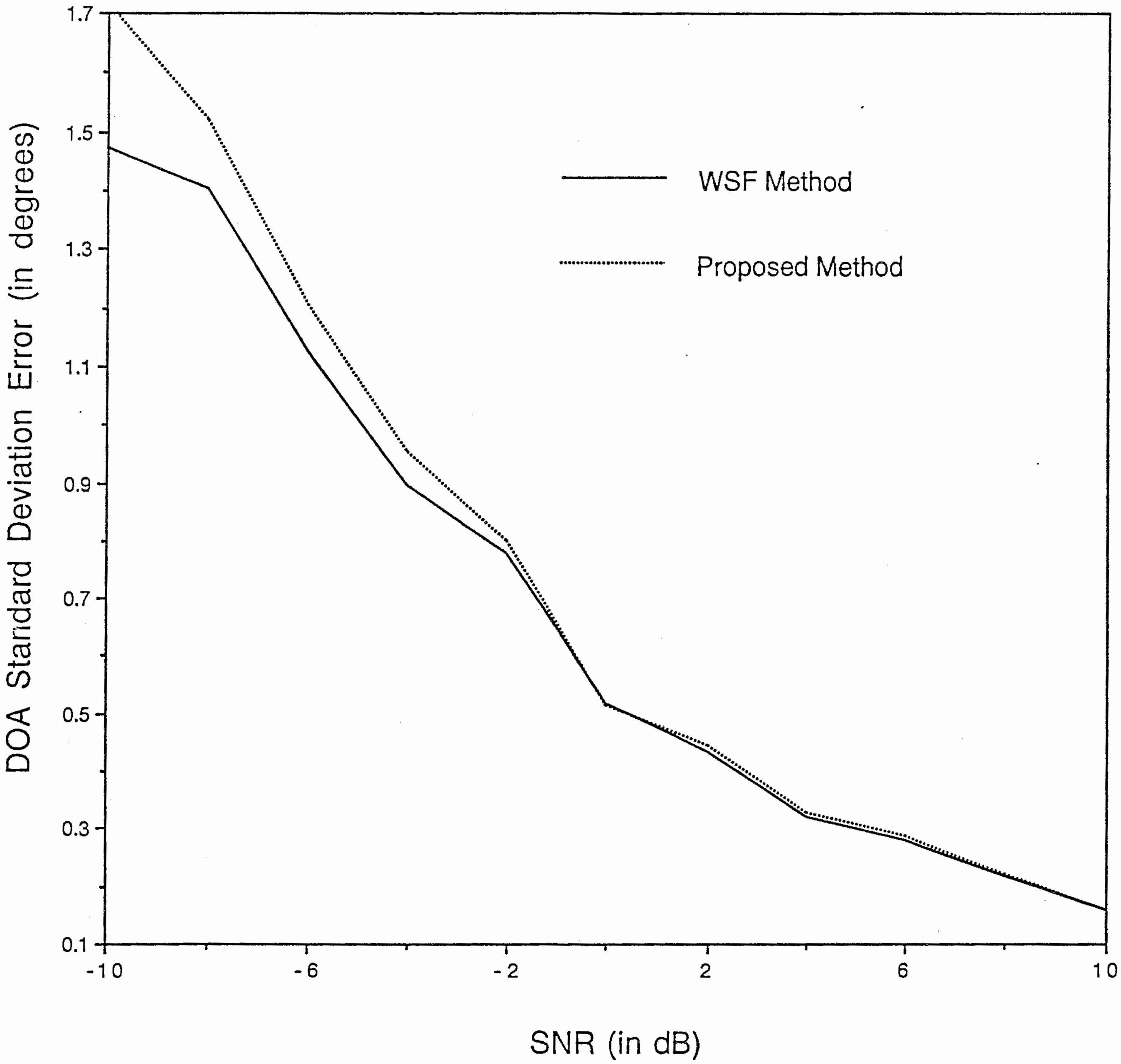


Figure 3(b) DOA Standard Deviation Error versus SNR for both WSF and proposed methods for coherent signals

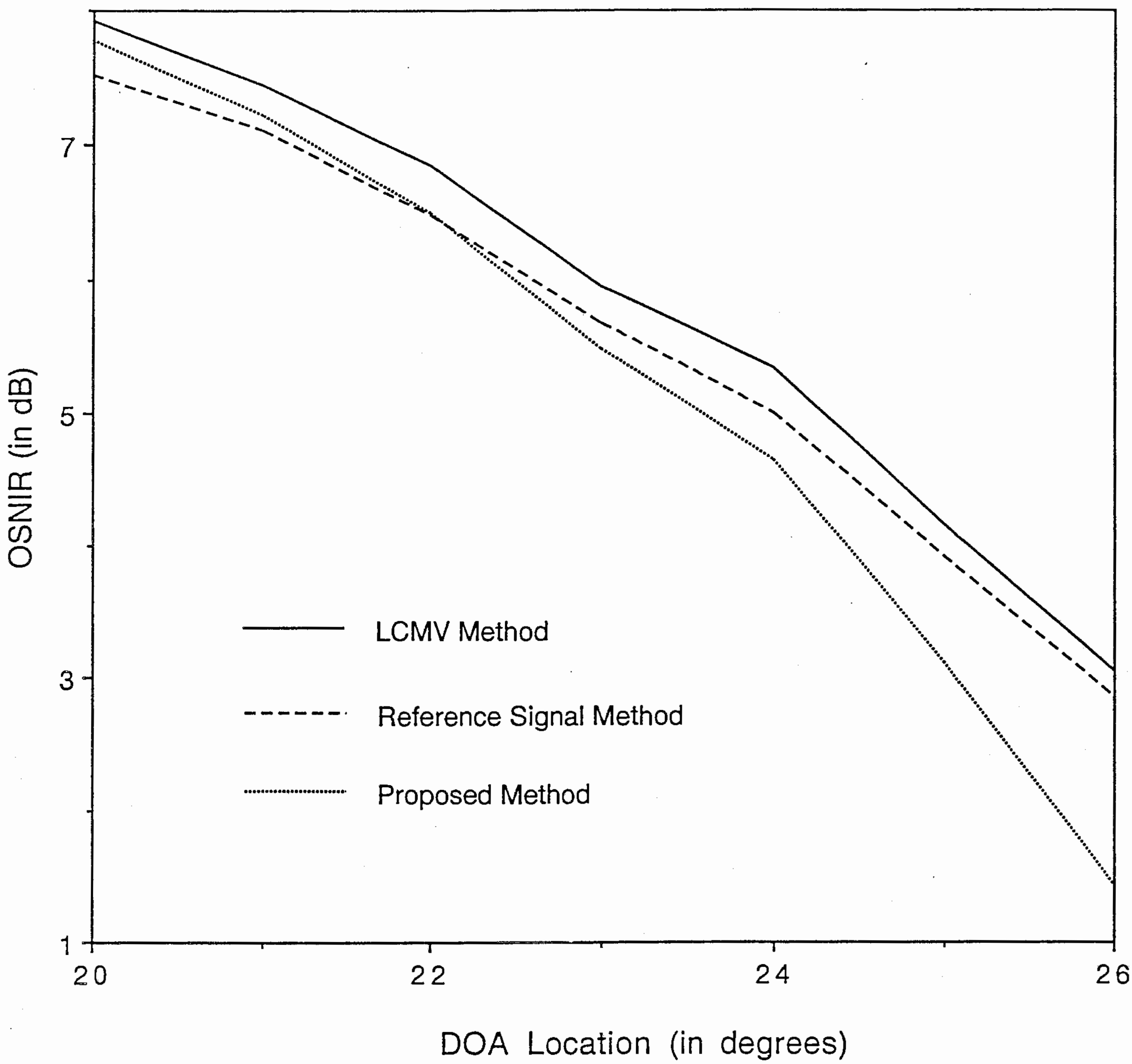


Figure 4(a) Values of OSNIR for various estimation methods versus location of one DOA (the nearest DOA at 30 degrees) SNR=0 dB

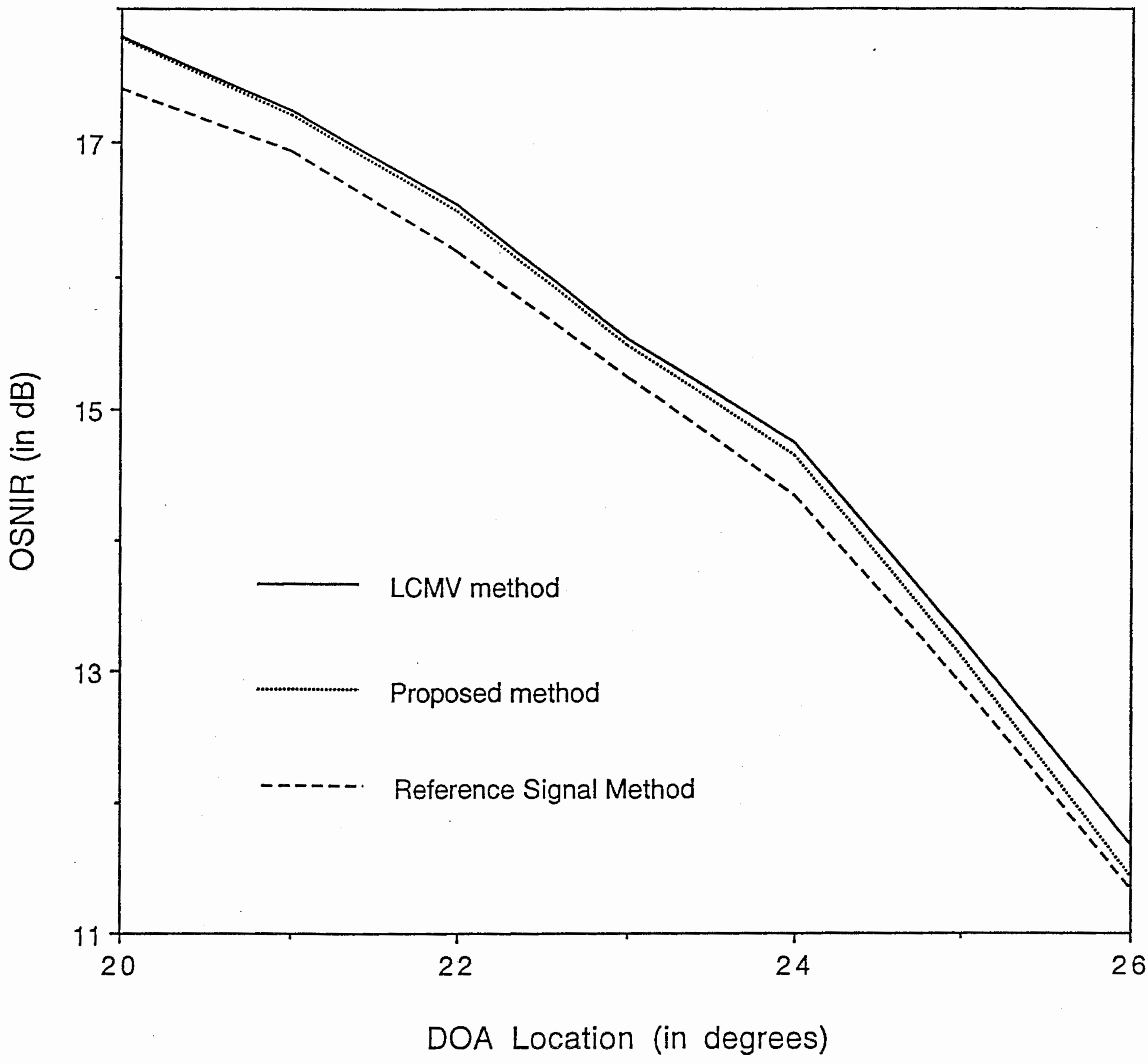
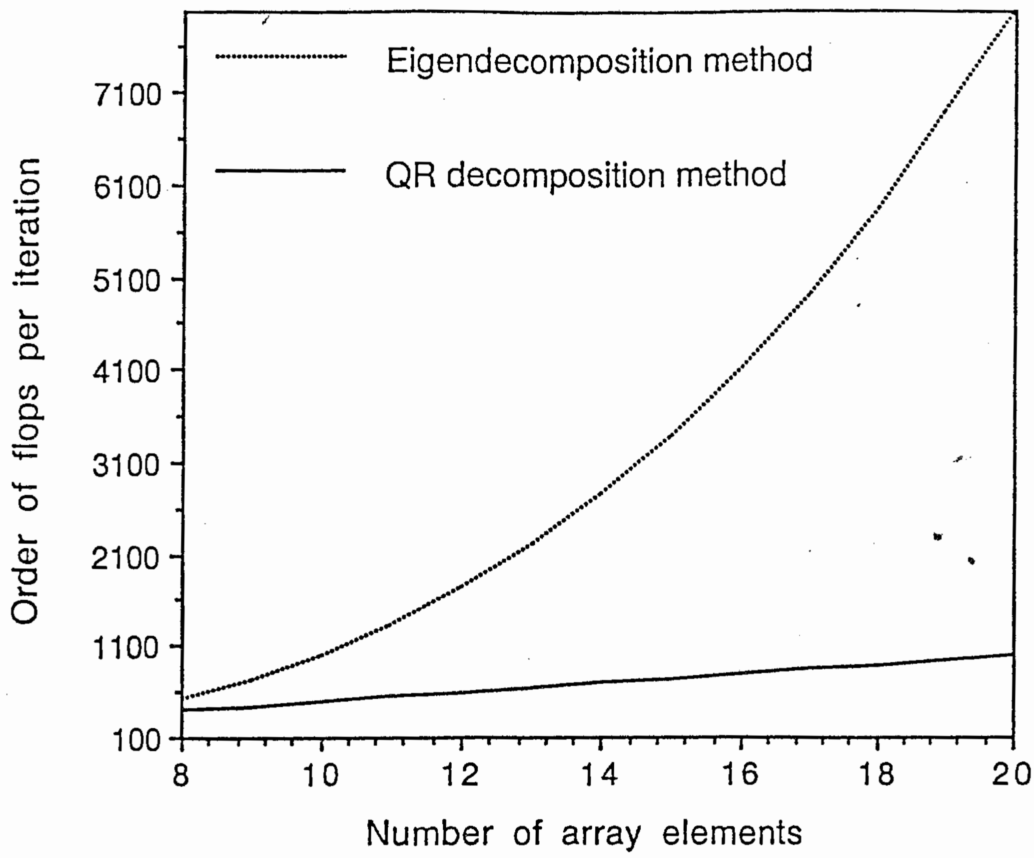
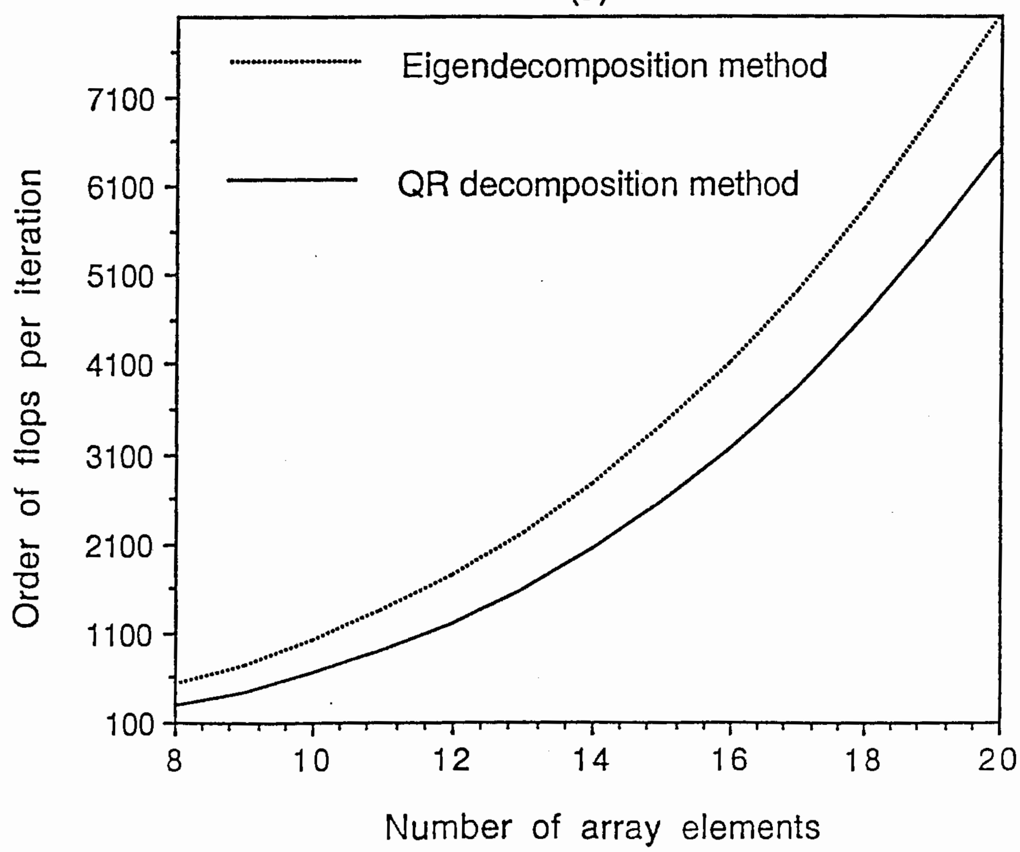


Figure 4(b) Values of OSNIR for various estimation methods versus location of one DOA (the nearest DOA at 30 degrees) SNR=10 dB



(a)



(b)

Figure 5 Rough comparison of the computational cost between the proposed and the eigendecomposition approaches
(a) d=7 and m between 8 and 20
(b) d=m-2 and m between 8 and 20