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Blueprint of a 3-Dimensional Vocal Fold Model

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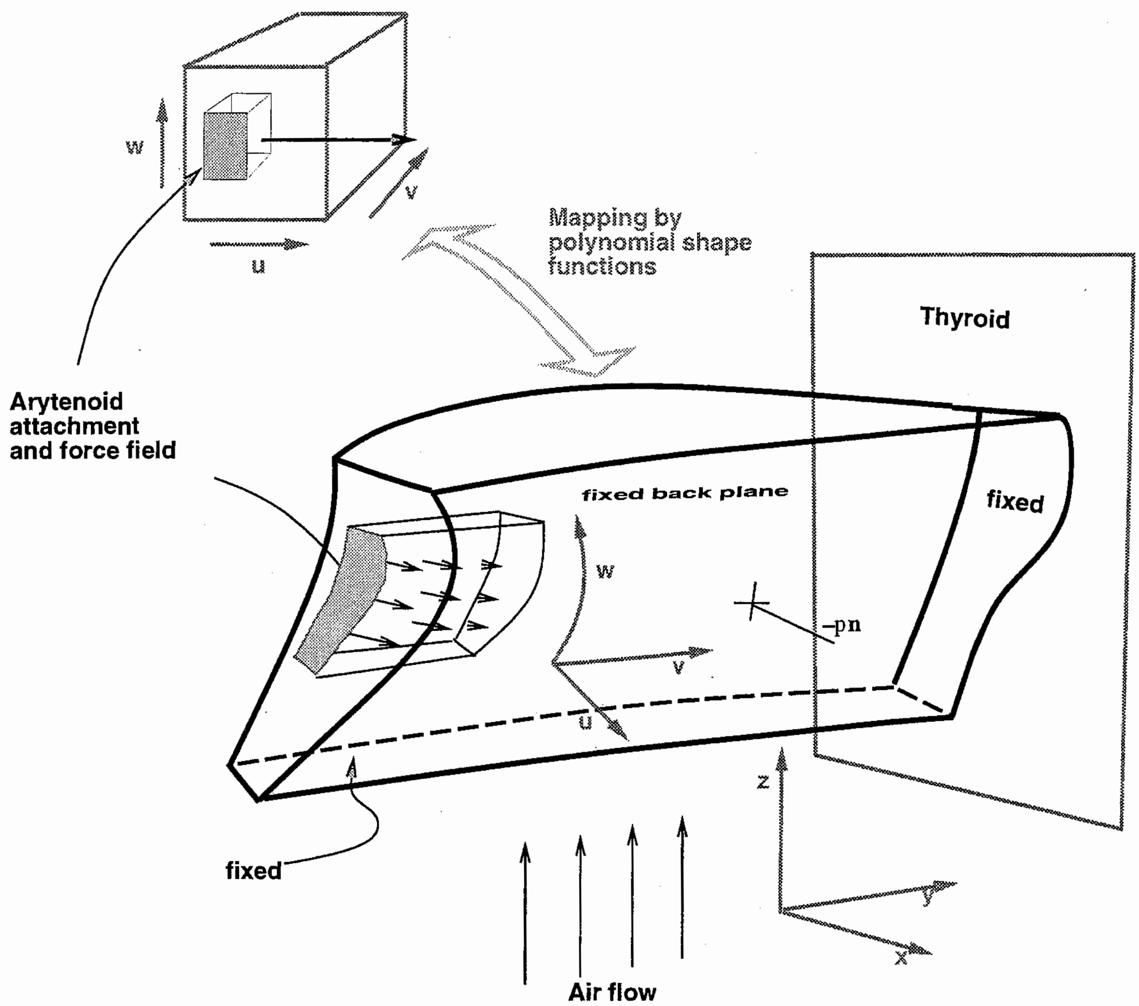


FIGURE 1. *Model overview. One of the two vocal folds is modeled, assuming complete asymmetry. The attachment of the arytenoids is modeled as a small volume in which a force distribution (force per volume) that balances with an externally applied force. The backplane, the upper and lower plane, and the plane of attachment to the thyroid cartilage are fixed. Deformation is achieved by applying a force that acts in the direction of the y axis, resulting in a force distribution as indicated by the arrow field. The air flow is in the direction of the z axis. The air pressure field enters as a traction force parallel to the surface normal.*

1. Introduction

This report attempts to cover the mathematical and continuum mechanical framework for the development of a 3-dimensional vocal fold model with air flow simulation and interaction between flow and vocal fold movement. Some implementation details are also given, as far as they could be worked out. The computational part of the model is not yet implemented.

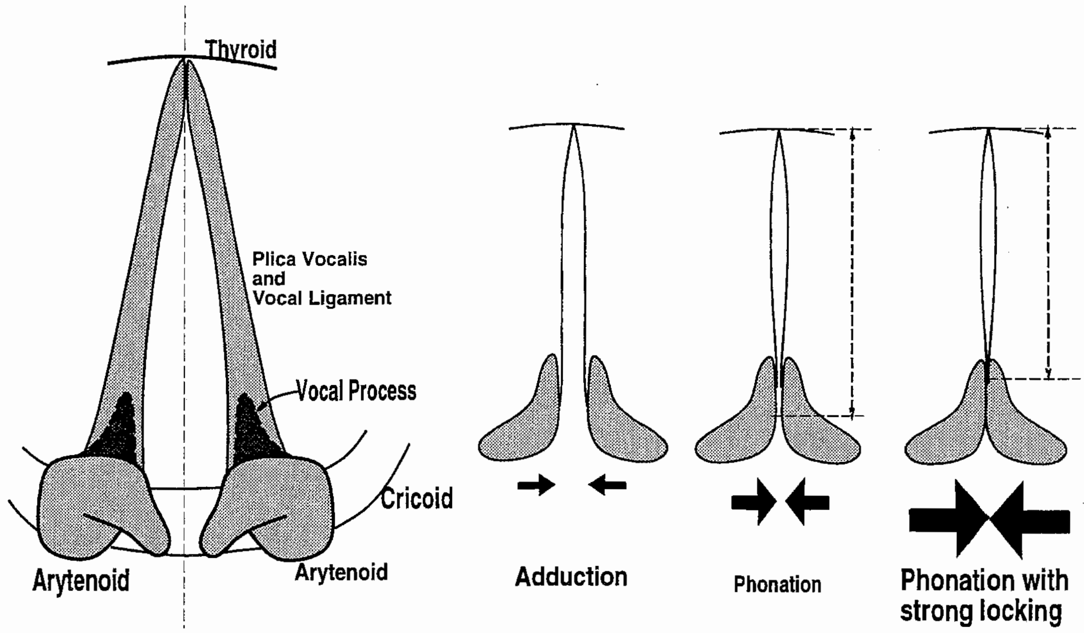


FIGURE 2. *The left side shows a view of the vocal folds from above. The anterior direction is up. The arytenoids are capable of sliding and rocking movements on the cricoid and may, by the actions of various muscles, abduct and adduct the vocal folds. The right side shows a suggested effect of various degrees of compression of the arytenoids. If loosely compressed, (see left and middle) the part of the vocal folds that may oscillate is longer than in the case of complete adduction and locking of the arytenoids towards each other.*

This model, as most current vocal fold models, is restricted to certain aspects of the larygeal sound production and articulation. From a physiological standpoint, the emphasis is on the articulatory function of the arytenoids. The arytenoids are the anterior attachment of the vocalis muscles and the articulation of the arytenoids substantially influences the oscillatory behavior of the vocal folds. One of the purposes of this modeling effort is to quantitatively investigate the effects of locking or releasing the arytenoids on the movements and oscillation of the vocal folds. The first part reports on the continuum mechanics of the vocal fold model. Much of the methods worked out there can be applied to other tissue deformation modeling. The second part concentrates on the air flow modeling.

2. Description of the vocal fold model

After presenting a simple vocal fold model that includes the action of the arytenoids in a simplifying manner, the continuum mechanical treatement of the moving vocal fold is described.

A basic overview of the model is shown in Fig. 1. In this model it is assumed that the vocal folds are symmetric and so only one of the two folds is contained in the model (the left one). Shown on the right is the attachment to the thyroid represented as a plane, and to the left a region in the vocal fold at which the arytenoids are attached. The action of the arytenoids is represented in a stylized manner by assuming a force field that is active in a portion of the vocal folds. The strength and (uniform) direction of this field is specified externally, and will later provide a way to combine the model with articulatory models whose scope is the entire larynx, see [**7**], or the whole vocal tract, see Dang and Honda [**5**]. The air flow between the vocal folds is assumed to be symmetric in spite of the problematics of this assumption, see discussion in the section on fluid mechanics.

The anatomy and biomechanical function is well presented in the literature, and in particular in the well illustrated book by Fink and Demarest [9]. The adduction and abduction of the vocal folds for phonation is mainly achieved by actions of the arytenoids using both a sliding and rocking movement of the arytenoids on the ridge of the crycoid cartilage. It is hypothesised that the effective length of the vocal folds is shortend if the two arytenoids are pressed together (see Fig. 2). This assumes that in a more relaxed configuration, the position of the arytenoids is not precisely fixed so they can undergo small oscillations. Further, the arytenoids are elastic bodies and therefore, the vocal processes to which the vocalis muscles are attached may participate in the oscillations of the vocal folds, while the main mass of the arytenoid is essentially resting. If the arytenoids are compressed completely and locked together (Fig. 2 on the right), the tissue around the arytenoids is prevented from oscillating. This amounts to effectively shortening the vocal folds. One goal of this modeling effort is to investigate the effects of the arytenoid articulation on the vocal folds' oscillatory patterns in a quantitative way.

3. Mathematical formulation of the model

For modeling the movements of the vocal folds it is convenient to assume a material coordinate system (Lagrangian description) that moves with the deformation and movements of the vocal folds. The body is described in terms of material coordinates, that is, each particle $\mathbf{p}$ is uniquely identified by a tuple (ξ, υ, ζ) . The particular choice of material coordinate system for the model is illustrated in Fig. 1 by a curvilinear coordinate system (See Appendix B). In the numerical implementation, this coordinate system will be established by using higher order polynomials as functions of ξ, υ, ζ. The general equations of motion of a deforming continuum, written in material coordinates, apply to this case:

$$\operatorname{div}(\boldsymbol{\sigma}) + f - \rho\dot{\mathbf{v}} = 0 \tag{3.1}$$

$\boldsymbol{\sigma}$ is the symmetric (Cauchy-) stress tensor, f are distributed body forces (N/m^3), ρ is the density of the material, $\mathbf{v}$ the velocity, and $\dot{\mathbf{v}} = \mathbf{a}$ the acceleration, both represented as material fields, that is, functions of the particle coordinates $\mathbf{p}$ and of time. The equations of motion above are only general and will be made specific further below. In particular, the relation between the deformation process and the stress tensor needs to be formally defined. In addition, initial conditions and (essential and natural) boundary conditions have to be specified. For the initial conditions, it may be assumed that the model is at rest at the time zero and has a certain configuration that will be refered to as its reference configuration, which will be represented by the symbol Ω_0. The motion, that is the solution of the equations of motion, is a time dependent mapping Φ from the reference configuration to the current configuration. In this model, the reference configuration may be some almost adducted state of the vocal folds. The essential boundary conditions are indicated in Fig. 1. Some of the surfaces are fixed, which is partially explained from the anatomy and partially from simple limitations of the domain of the model. The vocal folds are attached to the thyroid cartilage, which will be considered rigid and at rest. The other end of the vocal fold near the arytenoids is a free boundary but there is an imprinted force from the arytenoids, as explained further below. The back surface of the model and the upper and lower surfaces, as indicated in the figure are also assumed to be rigid and at rest. The currently proposed configuration and limitation of the model may change if it becomes necessary to extend the scope of the modeling. For example, the tissue around and between the arytenoids (to the left in the figure) is in reality a secondary opening of the vocal folds, sometimes called the glottal chink.

To avoid the complex computational problem of collision of the two vocal folds, the simpler problem will be considered first, namely that the two folds are moving in a symmetric way. In this case the collision problem is simplified to one in which the surface of the vocal fold model is constrained to move in one half-space. The collision modeling is reduced to a collision of a soft body with a rigid plane. In spite of this simplification, the more complicated problem of asymmetric collision can be addressed with the same formal method but is probably considerably more complicated in its practical implementation.

The movement of the vocal fold model is directly coupled with the dynamic behavior of the air flow between the vocal folds. The air flow simulation will be covered in the second part, starting with section 8. In general, the action of the air flow onto the vocal fold can be expressed as a rapidly varying non-uniform surface pressure. This pressure field enters the dynamic system vocal fold as part of the natural boundary conditions for the equations of motion.

4. Specification of the equations of motion

The principle of virtual work (actually virtual power, if $\delta\mathbf{v}$ is indeed understood as a velocity field) is employed to transform the equations of motion into a weak form that later becomes the starting point for a spatial discretization.

Let $\delta\mathbf{v}$ be an arbitrary velocity field that fulfilles the essential boundary conditions of the model, that is, it disappears where the model's movements are prescribed. The equation of motion is rewritten as a residual term that is supposed to be zero:

$$\mathbf{r} = \rho\dot{\mathbf{v}} - f - \operatorname{div}(\boldsymbol{\sigma}) = 0 \tag{4.1}$$

In what is known as weak formulation, it can be shown that the above is equivalent to demanding that, for any field $\delta\mathbf{v}$,

$$\delta w = \mathbf{r} \cdot \delta\mathbf{v} = 0 \tag{4.2}$$

The weak statement then is that for any field $\delta\mathbf{v}$ the integral over the body Ω_t in its current configuration Ω_t must disappear:

$$\delta W = \int_{\Omega_t} \mathbf{r} \cdot \delta\mathbf{v}\, \mathrm{d}\Omega_t = 0 \tag{4.3}$$

To outline in brevity the solution method, all fields are approximated by suitable linear superposition of a set of interpolation functions, so that the displacements, the velocities, and the virtual velocities can be represented as a sum of weighted functions, for example for the velocity field:

$$\mathbf{v} = \sum_a N_a \mathbf{v}_a$$

where the weight coefficients $\mathbf{v}_a$ are three-dimensional vectors. In the finite element method they are the node velocities[1].

The method proceeds by reformulating the virtual work equation with the approximated field variables and generating contributions to the virtual work equations. In the following terms of the virtual power integral will be investigated individually.

[1]Only in using the finite element method, the notation of nodal forces, nodal velocities and other nodal parameters, has a direct intuitive meaning as the properties of a body appear to be lumped to individual points that are connected in a mesh and have location, velocity, mass, force and other properties. Even though the concept of coefficients is more general and also covers the case of spectral coefficients, it is useful to always think in terms of nodal forces, displacements and velocities.

4.1. Inertia and mass matrix.

The kinetic energy rate term in the virtual work integral associated with the acceleration is:

$$\delta W_m = \int_{\Omega_t} \rho \dot{\mathbf{v}} \cdot \delta \mathbf{v} =$$

A representation by weight functions $\delta \mathbf{v} = \sum_a N_a \delta \mathbf{v}_a$ and $\dot{\mathbf{v}} = \sum_b N_b \dot{\mathbf{v}}_b$ results in a contributions

$$\delta W_m = \int_{\Omega_t} \rho (\sum_b N_b \dot{\mathbf{v}}_b) \cdot (\sum_a N_a \delta \mathbf{v}_a) = \sum_{ab} (\int_{\Omega_t} \rho N_b N_a) \dot{\mathbf{v}}_b \cdot \delta \mathbf{v}_a \tag{4.4}$$

This can be written in components:

$$\delta W_m = \sum_a \sum_b (\int_{\Omega_t} \rho N_b N_a) \dot{\mathbf{v}}_{bi} \delta \mathbf{v}_{ai} \quad = M_{ab} \dot{\mathbf{v}}_{bi} \delta \mathbf{v}_{ai}$$

(Summation over multiple indices is implied.) Since the $\delta \mathbf{v}_a$ are arbitrary, the lumping of inertia effects is manifest as forces

$$F_a^{\text{kin}} = \sum_b M_{ab} \dot{\mathbf{v}}_b \tag{4.5}$$

The symmetric mass matrix M transforms accelerations into inertial forces (which act on the nodes, if a finite element implementation is used).

Remark: It should be pointed out that the mass matrix actually changes over time. This can be seen when it is computed once over the reference configuration and then over the current configuration. The actual integration is always executed over the domain $d\Omega_M$ of material coordinates that parameterizes the volume of the body, no matter if it is in its reference or in its current configuration. When evaluating the integral over the reference configuration Ω_0, the Jacobian J_0 of the mapping between $d\Omega_M$ and Ω_0 must be used to weight the volume element of the configuration $d\Omega_M$: $d\Omega_0 = J_0 d\xi d\upsilon d\zeta$. Since Ω_0 does not change, could we just compute the mass matrix once by integrating over Ω_0? Strictly speaking that is not allowed, since the definition of the mass matrix requires integration over Ω_t, as shown above. However, in this context we are dealing with almost incompressible tissue with approximately constant density. So, if we stretch the truth a little and assume that we are able to **strictly** enforce the condition of incompressibility (including in the numerical implementation) then the simulated movement of the body will be isochoric and the Jacobian J of the mapping from Ω_0 to Ω_t will be always 1 everywhere. Under this assumption, the mass matrix computed over the current configuration volume is equal to the one computed over the reference. So it is for convenience to assume that this approximation is valid:

$$M_{ab} = M_{ab}^0 = \rho \int_{\Omega_0} N_a(\mathbf{p}) N_b(\mathbf{p}) d\Omega_0 = \rho \int_{d\Omega_M} N_a(\xi, \upsilon, \zeta) N_b(\xi, \upsilon, \zeta) J_0(\xi, \upsilon, \zeta) d\xi d\upsilon d\zeta \tag{4.6}$$

This way, since M is computed only when the reference configuration changes, the computation has to occure only once, and a linearization is not necessary.

4.2. Action of the arytenoids modeled as a body force term .

Forces such as Coriolis and gravity are contained in the following component of the virtual work:

$$\delta W_f = \int_{\Omega_t} f \cdot \delta \mathbf{v} \, d\Omega_t$$

f is a force density in units of Newton per m^3 or dyne/cm^3 and may vary over the volume. It may or may not depend on the movement.

It will be assumed that Coriolis and gravitational forces can be neglected, but in this model there is another use of the body force term f. It will be used to introduce the forces by which the arytenoid acts on the vocal folds. This is admittedly a kluge. More physically correct - but probably more computational intense - modeling would represent the arytenoid as an elastic structure whose movements are modeled as a dynamic system which interfaces with the vocal folds' dynamic system. It can be expected that adding a rigid structure that interfaces with the vocal fold (and thus prescribes the movement of the nodes on the interface) would result in a stiffer behavior of the tissue in the vicinity of the arytenoid. However, since we would like to describe the movements of the vocal fold with sufficiently low order polynomials, this reduction of flexibility might result in too much artificial stiffening.

The proposed "non-physical" modeling is illustrated in Fig 1. as small partial volume, corresponding to the material coordinates in the brick formed by the intervals, $A = [\xi_1, \xi_2] \times [0, \upsilon_0] \times [\zeta_1, \zeta_2]$. In this region, the body forces are defined that represent the action of the arytenoids.

One has to make a reasonable assumption about the distribution of this force field. In any case, we need to be able to compute with ease a total force that balances the force field, since this total force will be an externally controlled parameter. The simplest assumption, that the force field is uniform throughout the region, seems a bit unrealistic since the tip of the vocalis process certainly exerts a much smaller force in x direction than the point at which the arytenoid enters the vocal fold. Another possible assumption is to specify a torque at an (artificially introduced) pivot of the arytenoid and to compute the resulting force density. This appears to add too much complexity while being still inaccurate, since the movements of the arytenoids can be both sliding and rocking movements.

A short consideration about the force density distribution: Consider a pivot point o outside the vocal fold element. The pivot axis may for simplicity be parallel to the z-axis, so things are planar. Let s denote the distance from the pivot to a point x in the vocal fold projected onto the xy-plane. If the force is acting perpendicular to the connecting line x to o, the torque is $\tau = f(x)s(x)$. If the transmitting element behaves like a flexible beam the force itself depends on the distance s in a monotonically falling manner. In particular, if we are dealing with a uniform beam, we would get f proportional to $1/s^2$. Since the arytenoid is thinner at the vocal process end than near the entrance point, this extreme seems as unlikely as the assumption of a constant force. An intermediate assumption could be using $f \sim 1/s$.

So in the most primitive representation of the arytenoids, it will be assumed that the force field is unidirectional and can be described by a scalar weighting function w times a vector f_{aryt} that is controlled from outside:

$$f(\mathbf{x}) = w(\mathbf{x}) f_{\text{aryt}}$$

The externally controlled parameter will be denoted T_{aryt}. It is a vector parallel to f_{aryt}. To specify f_{aryt} we need to compute:

$$T_{\text{aryt}} = \int_{\Omega_t} f(x) d\Omega_t = f_{\text{aryt}} \int_{\Omega_t} w(\mathbf{x}) d\Omega_t$$

and thus

$$f_{\text{aryt}} = T_{\text{aryt}} / \int_{\Omega_t} w(\mathbf{x}) d\Omega_t$$

Since this would result in the necessity to recompute f_{aryt} every time the system changes (since $d\Omega_t = J_0 J d\xi d\upsilon d\zeta$, and J, the Jacobian, changes with the deformation), it is advantageous to define the force field either on the reference system (with $d\Omega_0 = J_0 d\xi d\upsilon d\zeta$) and most simply directly in material coordinates. This would result in:

$$f_{\mathrm{aryt}} = T_{\mathrm{aryt}} / \int_{\Omega_0} \mathrm{w}(\mathbf{p}) d\Omega_0$$

It is now necessary to compute the virtual work contribution of these forces. This will allow the force field to be lumped to the nodes by projecting it on the shape functions of each node. The expression for the virtual work field,

$$\delta W_f = \int_{\Omega_t} f \cdot \delta \mathbf{v} d\Omega_t$$

gives rise to an internal force associated with node a of

$$F_a^{\mathrm{int}} = \int_{\Omega_t} f(\mathbf{x}) N_a(\mathbf{x}) d\Omega_t = f_{\mathrm{aryt}} \int_{\Omega_t} \mathrm{w}(\mathbf{x}) N_a(\mathbf{x}) d\Omega_t \quad . \tag{4.7}$$

Obviously, the computation of the resulting node forces depends on the current configuration, no matter how f_{aryt} is specified. To simplify the computation it could be assumed that the movement is isochoric, which means that $J \approx 1$ everywhere. In this case, the components F_a^{body} can be precomputed as follows:

$$F_a^{\mathrm{int}} = T_{\mathrm{aryt}} \frac{\int_{\Omega_0} \mathrm{w}(\mathbf{p}) N_a(\mathbf{p}) d\Omega_0}{\int_{\Omega_0} \mathrm{w}(\mathbf{p}) d\Omega_0} \tag{4.8}$$

These integral values can be precomputed once the reference configuration is specified.

4.3. Stress field.

The stress field gives rise to a virtual work contribution (stress power):

$$\delta W_S = \int_{\Omega_t} \mathrm{div}\,(\boldsymbol{\sigma}) \cdot \delta \mathbf{v}\, d\Omega_t \quad .$$

Using the identity,

$$\mathrm{div}\,(\boldsymbol{\sigma}) \cdot \delta \mathbf{v} = \mathrm{div}\,(\boldsymbol{\sigma} \delta \mathbf{v}) - \boldsymbol{\sigma} : \nabla\,(\delta \mathbf{v})$$

and Gauss' integration theorem, it can be shown that it is ($\mathbf{n}$ representing the surface normal):

$$\begin{aligned} \delta W_S &= \int_{\partial\Omega_t} \mathbf{n} \cdot (\boldsymbol{\sigma}) \delta \mathbf{v} d\mathrm{A}_t - \int_{\Omega_t} \boldsymbol{\sigma} : \nabla\,(\delta \mathbf{v})\, d\Omega_t \\ &= \int_{\partial\Omega_t} (\boldsymbol{\sigma} \mathbf{n}) \cdot \delta \mathbf{v} d\mathrm{A}_t - \int_{\Omega_t} \boldsymbol{\sigma} : \nabla\,(\delta \mathbf{v})\, d\Omega_t \end{aligned}$$

According to Cauchy's theorem of the existence of stress (see Gurtin, page 101ff), the stress field at the surface must be balanced with a surface traction that acts on the surface. It is obtained, on the surface: $\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}$. Further, since $\boldsymbol{\sigma}$ is symmetrical, $\nabla\,(\delta \mathbf{v})$ can be replaced by its symmetric part $\delta D = \frac{1}{2}(\nabla\,(\delta \mathbf{v}) + \nabla\,(\delta \mathbf{v})^T)$ (see DEF. 5 and Lemma 1). The above can then be rewritten as:

$$\delta W_S = \delta W_t - \delta W_{\boldsymbol{\sigma}} = \int_{\partial\Omega_t} \mathbf{t} \cdot \delta \mathbf{v}\, d\mathrm{A}_t - \int_{\Omega_t} \boldsymbol{\sigma} : \delta D\, d\Omega_t \tag{4.9}$$

The surface traction will be discussed in the next subsection. Here will be dealt with the other term:

$$\delta W_{\boldsymbol{\sigma}} = \int_{\Omega_t} \boldsymbol{\sigma} : \delta D \, \mathrm{d}\Omega_t$$

Obviously, it is necessary to compute the stress $\boldsymbol{\sigma}$ as a function of the kinematic variables and other parameters to make the model complete.

In general, the stress tensor $\boldsymbol{\sigma}$ depends on both the deformation and the rate of deformation tensors. In this model, it will be assumed that the dependency can be decomposed into a purely hyper-elastic term and a viscous term. The hyper-elastic term will be discussed first. For a good presentation of most of these issues, see the book [**3**].

Considering Lemma 1 (see Appendix C), the definition of hyperelastic materials is facilitated by introducing the second Piola stress tensor which is a push-back version of the Cauchy stress tensor. The relation between the two is by the Piola-transformation:

$$\boldsymbol{\sigma} = J^{-1} F S F^T \quad , \quad S = J F^{-1} \boldsymbol{\sigma} F^{-T} \tag{4.10}$$

For the description of hyperelastic materials, the 2nd Piola tensor can be obtained as the total differential of an energy density function which is a function of the Cauchy deformation tensor: $\Psi(C)$:

$$S = 2\frac{\partial \Psi(C)}{\partial C} \quad , \quad S_{ij} = 2\frac{\partial \Psi(C)}{\partial C_{ij}}$$

As shown in Appendix C, for the case of incompressible materials a requirement emerges that the 2nd Piola stress must be modified, in that it now depends on a pressure field that is not a function of the Cauchy tensor C. This is also true for the case that the 2nd Piola tensor contains other terms that are unrelated to the Cauchy tensor. In the present case there is an additional term that results from viscous shear damping, and is therefore some function L of the velocity gradient D, or its material push back, the rate of the Euler tensor: $\dot{E} = F^T D F$. So the 2nd Piola tensor becomes:

$$S = 2\frac{\partial \Psi(C)}{\partial C} + L[\dot{E}] + \gamma J C^{-1}$$

For an incompressible tissue the stress-strain relations usually assume that the tissue deformes isochorically. In fact, if the stress strain relations were obtained in experiments, the empirically found relations are only valid if the movements are isochoric, since incompressible tissue can only deform isochorically. In a numerical implementation, it is usually not possible to realize exactly ischoric movements. One way to get around the resulting problems is to use a modified Cauchy tensor that has always a determinante of 1, namely

$$\overline{C} = J^{-\frac{2}{3}} C$$

and rewrite the strain energy density function $\Psi(C)$ by $\overline{\Psi}(C) := \Psi(\overline{C})$. A modified 2nd Piola tensor can be formulated by writing:

$$S = 2\frac{\partial \Psi \overline{C}}{\partial \overline{C}} \frac{\partial \overline{C}}{\partial C} = \overline{S} \frac{\partial \overline{C}}{\partial C} \quad \text{with} \quad \overline{S} := 2\frac{\partial \Psi \overline{C}}{\partial \overline{C}}$$

S is then obtained via the following operation (see Appendix for a derivation):

$$S = J^{-2/3} \left(\overline{S} - \frac{1}{3} \left(\overline{S} : C \right) C^{-1} \right) \tag{4.11}$$

It can be seen (see appendix) that the modified elasticity potential $\Psi(\overline{C})$ results in a stress field in which the pressure term disappears.

4.3.1. *Viscous term.* The viscous stress contribution will be assumed to be simply:

$$\boldsymbol{\sigma}_v = 2\mu D \quad .$$

Since a pressure term is undesirable in this equation, it must be removed and only the deviatoric part used:

$$DEV(\boldsymbol{\sigma}_v) = \boldsymbol{\sigma}_v - \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}_v I = 2\mu D - \frac{2\mu}{3}\mathrm{div}\,(v)\,I$$

This is done anticipating that the dynamic constraint of isochoric movement, $\mathrm{div}\,(v) = 0$, can not be enforced strictly when using a discrete approximation.

4.3.2. *Discretization.* The discretization of the stress terms is achieved by again making use of the discretization of the virtual velocity field $\delta\mathbf{v}$ and its spatial gradient (see A.9). Using $\nabla\delta\mathbf{v} = \sum_a \delta\mathbf{v}_a \otimes \nabla N_a$,

$$\int_{\Omega_t} \boldsymbol{\sigma} : \nabla\delta\mathbf{v}\mathrm{d}\Omega_t = \sum_a \left(\int_{\Omega_t} \boldsymbol{\sigma}\nabla N_a \mathrm{d}\Omega_t \right) \delta\mathbf{v}_a \tag{4.12}$$

is obtained. Therefore, the resulting node force is:

$$F_a^{\sigma} = \int_{\Omega_t} \boldsymbol{\sigma}\nabla N_a \mathrm{d}\Omega_t \tag{4.13}$$

4.4. Surface traction.

The surface traction in this model is due to the external pressure field which varies over the surface of the vocal fold. As such it is a function of the current location of a point on the surface and its direction is always in the negative surface normal. So if $p(\mathbf{x})$ is the pressure then a contribution to the virtual work is obtained:

$$\delta W_t = \int_{\partial\Omega_t} \mathbf{t}\cdot\delta\mathbf{v}\,\mathrm{dA}_t = \int_{\partial\Omega_t} p(\mathbf{x})\mathbf{n}(\mathbf{x})\cdot\delta\mathbf{v}\,\mathrm{dA}_t$$

In considering the variation of node a only, the lumped force is obtained:

$$F_a^{\mathrm{surf}} = \int_{\partial\Omega_t} p(\mathbf{x})\mathbf{n}(\mathbf{x})N_a(\mathbf{x})\,\mathrm{dA}_t \quad . \tag{4.14}$$

Practical note: The computation of this surface integral is usually achieved by numerical means. First a parametric description of the surface is needed. For that two variables that parameterize the surface, α and β are used. The shape functions are restriced to the surface so that formally functions $N_a(\alpha,\beta)$ are obtained[2]. The surface normal is

$$\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{x}(\alpha,\beta)) = \frac{\frac{\partial\mathbf{x}}{\partial\alpha}\times\frac{\partial\mathbf{x}}{\partial\beta}}{|\frac{\partial\mathbf{x}}{\partial\alpha}\times\frac{\partial\mathbf{x}}{\partial\beta}|}$$

and the surface area is:

$$\mathrm{dA}_t = \left|\frac{\partial\mathbf{x}}{\partial\alpha}\times\frac{\partial\mathbf{x}}{\partial\beta}\right| d\alpha d\beta \quad .$$

So the normalization term for the surface normal drops out:

$$\mathbf{n}dA = \frac{\partial\mathbf{x}}{\partial\alpha}\times\frac{\partial\mathbf{x}}{\partial\beta}d\alpha d\beta \tag{4.15}$$

[2]The same symbol N_a is used for both the surface shape functions and the three-parameter shape functions for the interior of the region. Since the three-parameter shape functions are usually outer products of one-parametric functions, one of the three variables may be simply set to a fixed value to achieve the surface parametrization.

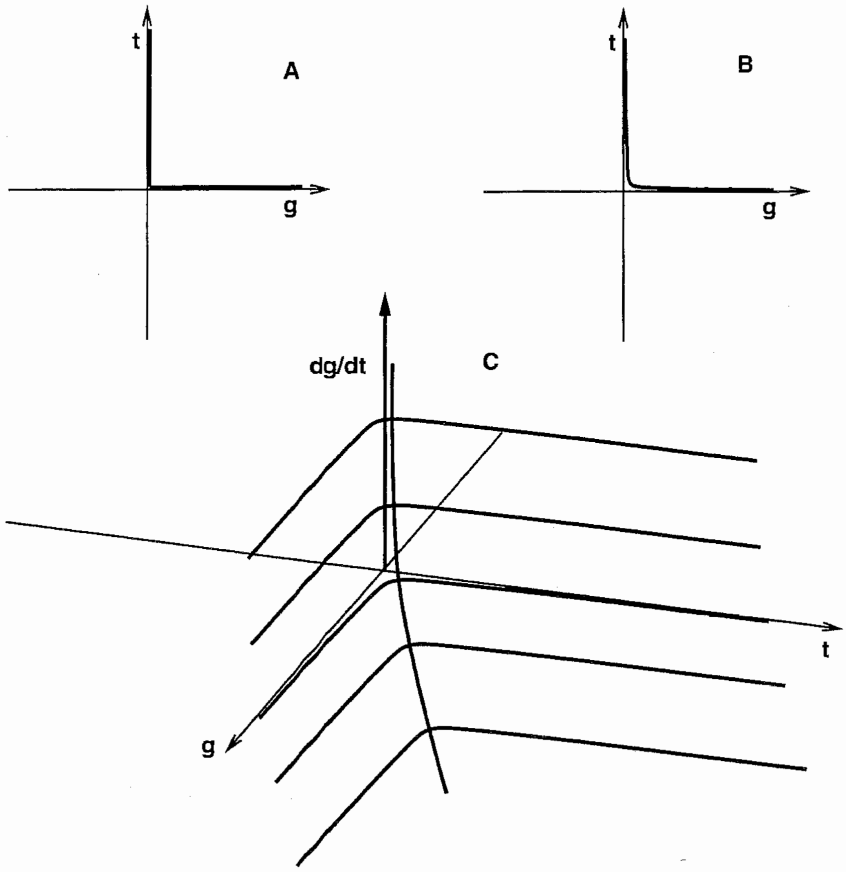


FIGURE 3. *A: Idealized constraint. B: Numerically realistic approximation of the constraint. C: Constraint depending on gap velocity*

In the appendix (see 14.1) the case of a constant surface pressure is considered, leading to a number of coefficients that can be precomputed. However, in the general case it is better to proceed along the lines of Gauss-Legendre integration: For each Gauss point with weight w_g in the domain, that is, at a parameter value (α_g, β_g),

$$\text{calculate} \quad \mathbf{x} = \sum_a N_a \mathbf{x}_a \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial \alpha} = \sum_a \frac{\partial N_a}{\partial \alpha} \mathbf{x}_a \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial \beta} = \sum_a \frac{\partial N_a}{\partial \beta} \mathbf{x}_a$$

Then compute the vector $\mathbf{y}_g = \frac{\partial \mathbf{x}}{\partial \alpha} \times \frac{\partial \mathbf{x}}{\partial \beta}$ and find the pressure p at the location $\mathbf{x}$.

$$F_{t,a} = \sum_g w_g p(\alpha_g, \beta_g) N_a(\alpha_g, \beta_g) \mathbf{y}_g$$

4.5. Collision constraint.

Only one half of the glottis is represented in the model, assuming symmetry. Therefore, the movement of the one vocal fold is constrained to one half space. The symmetry assumption implies that the collision with the opposite fold always occures exactly at the midline and therefore provides precisely the collision forces necessary to stop the left vocal fold (which is modeled) exactly at the midline. So the collision that must be modeled is a collision without friction between a soft body and the ridid y-z plane at x=0. The gap $g(\mathbf{x})$ is defined as

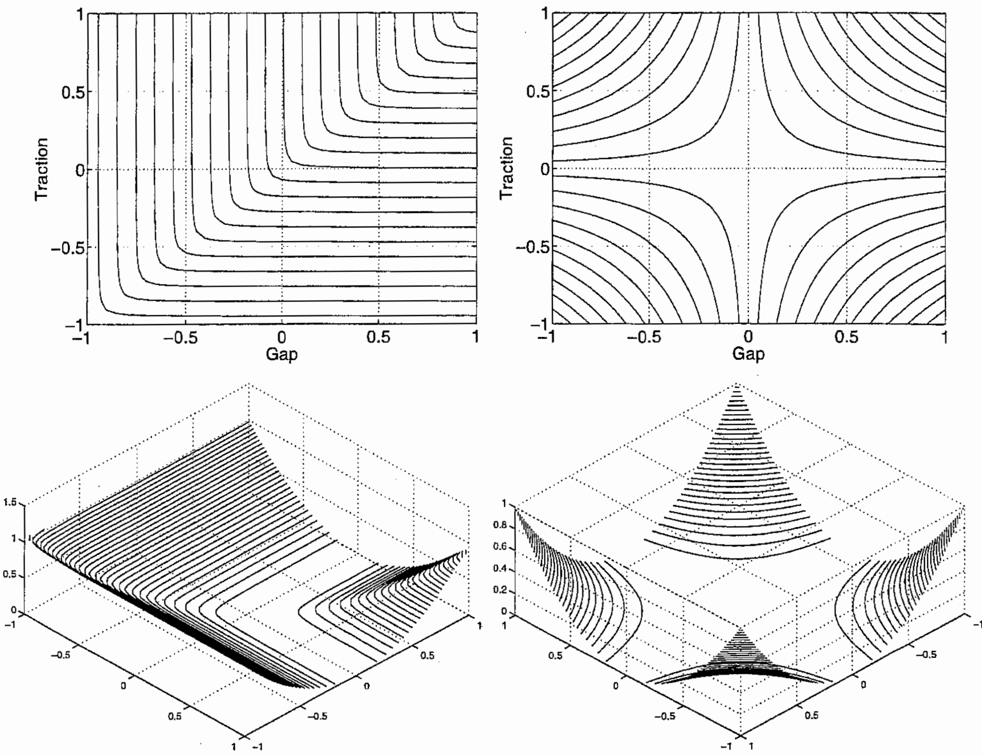


FIGURE 4. *Contour landscapes of the squared constraint equations* $w(g,\tau) = \frac{g+\tau}{2} - \sqrt{\left(\frac{g-\tau}{2}\right)^2 + \epsilon}$ *(left) and* $g\tau - \epsilon$ *(right)*

the distance from a surface point of the vocal fold $\mathbf{x}$ to the constraint plane. In this case, of course, $g(\mathbf{x}) = -\mathbf{x}_1$. The traction that will keep the gap from becoming negative is a Lagrange multiplier variable that must be zero as soon as there is no contact. The traction acts in negative x-direction in this model's case, and its magnitude is denoted $\tau(\mathbf{x})$. Hence, the following conditions apply:

$$g \geq 0 \quad , \quad \tau \geq 0 \quad \text{, and} \quad g\tau = 0 \tag{4.16}$$

The last equation means that either g or τ or both must be equal zero. The domain on which g and τ can be is shown in Fig. 3-A as thick lines. It is numerically impossible to implement such a constraint and so an approximation could be

$$g\tau - \epsilon = 0$$

However, one small problem is that the function $f(g\tau) = g\tau - \epsilon$ has a saddle point at the origin. In the book by Bathe on Finite Elements [**2**] a different function is proposed:

$$w(g,\tau) = \frac{g+\tau}{2} - \sqrt{\left(\frac{g-\tau}{2}\right)^2 + \epsilon}$$

It turns out that $w(g,\tau) = 0$ implies $g\tau = \epsilon$. The advantage of this function can be seen when we compare the landscape of the square of this function with the square of the function $g\tau - \epsilon$, see Fig. 4.

If the constraint is fulfilled the force will be $\tau = \epsilon/g$. From a numerical standpoint it may be benificial to make the small number ϵ in the constraint depending on the gap velocity: If the gap velocity is negative, that is, the body is moving towards the boundary, we would like a bit more repelling force "early on". Thus, ϵ should increase with the speed of approach and also have the effect of flattening the constraint curve so that the counter force begins to act already at a larger gap. With this in mind the choice is:

$$\epsilon(\dot{g}) = \epsilon_0 \exp(\alpha \dot{g})$$

This results in a constraint function:

$$w(g, \dot{g}, \tau) = \frac{g+\tau}{2} - \sqrt{\left(\frac{g-\tau}{2}\right)^2 + \epsilon_0 \exp(\alpha \dot{g})} \tag{4.17}$$

The resulting constraint surface $w(g, \dot{g}, \tau) = 0$ is sketched in Fig. 3-C.

In the non-linearized form, the inclusion of the constraint amounts to adding a term $\tau\mathbf{m}$ to the traction in the virtual work of the surface forces, where $\mathbf{m}$ represents the direction vector in which the collision force acts (in this model a unit vector in the negative x-direction). So the virtual work term related to the surface traction is:

$$\delta W_t = \int_{\partial\Omega_t} \tau\mathbf{m} \cdot \delta\mathbf{v}\, \mathrm{dA}_t \tag{4.18}$$

To find τ the constraint equation $w = 0$ must be solved. This is trivial because $w = 0$ implies that τ can be computed as $\tau = \epsilon_0 \exp(\alpha\dot{g})/g$.

It may be asked why not just use $w = \tau g + \epsilon(\dot{g})$? The significance of the more complicated function $w(g, \dot{g}, \tau))$ can only be seen when considering linearization. The negative gradient of the function is defined everywhere on the (g, τ) plane (for any $\dot{g}$) and points always directly to the next point on the constraint curve $w = 0$, where it disappears. This thread will be picked up again later.

The surface traction field that enforces the impenetrability condition is different from zero where the vocal folds are in contact (that is, numerically very close to contact). An integration over the contact area is required to lump these surface tractions to node forces. Formally this is achieved, for node a, by the surface integral

$$F_a^{\mathrm{col}} = \int \mathrm{d}\Omega_t \tau(\mathbf{x})\mathbf{m}N_a \mathrm{dA}_t \tag{4.19}$$

In the implementation, the contact area will be covered by several test points for each of which the contact condition is evaluated. It is reasonable to simply use a large number of Gauss integration points on the surface of the vocal fold and use Gauss-Legendre integration. The surface integral (4.19) is then computed as a weighted sum of the pointwise evaluated integrant.

5. Augmented Virtual Work equation

Now we should collect the pieces again and put them together into one account. It is convenient to follow some methods described nicely in the book by Bonet and Wood [**3**]. However, it is necessary to extend it a bit to the case of dynamic systems and systems with collision which are not covered there.

The solution to the equations of motion is represented by the symbol Φ. It describes the mapping from a reference configuration to the current configuration. The change of this mapping constitutes the movement that is the solution of the equations of motion together with appropriate boundary conditions and constraints.

Since the vocal folds are not really elastic and have dissipative losses, it is not possible to formulate the same variational statements that hold true for elastic systems, in which the strain energy can be derived from an elasticity potential. Even though this is used to describe the *elastic part* of the stress tensor, it is no longer possible to employ an elastic potential whose derivative is the stress tensor.

In elastic systems constraints can be incorporated related to the deformation by augmenting the strain energy function by a functional that is chosen such that it disappears when the constraint is exactly fulfilled. This method is described in detail in the book by Bonet and Wood (and in other places). The method can be described as follows: The total energy function of the system is written down and augmented by the constraints, then the total variation is taken to obtain an augmented virtual work equation. In implementing the incompressibility constraint, the central idea is to separate the volume changing component of the strain energy and the distortional strain energy so that the stress tensor $\boldsymbol{\sigma}$ does not contain any volumetric components. Since it is not possible to write one strain energy function whose differential is the stress tensor, a virtual work equation is obtained by multiplying the equations of motion with an arbitrary velocity field and integrating. This functional is augmented by the total variation of another functional that takes care of the constraints.

It is assumed that some solution to the equations of motion exist, and it is at time t a mapping Φ from a reference configuration to the current configuration. In this configuration the equations of motion are satisfied. So a weak statement that holds true for any virtual displacement or velocity field $\delta\mathbf{v}$ becomes the the virtual work equation:

$$\delta W(\Phi, \delta\mathbf{v}) = \int_{\Omega_t} \rho\dot{\mathbf{v}} \cdot \delta\mathbf{v} - \int_{\Omega_t} f \cdot \delta\mathbf{v}\, \mathrm{d}\Omega_t + \int_{\Omega_t} \boldsymbol{\sigma} : \nabla\left(\delta\mathbf{v}\right) \mathrm{d}\Omega_t - \int_{\partial\Omega_t} \mathbf{t} \cdot \delta\mathbf{v}\mathrm{dA}_t = 0 \tag{5.1}$$

In this equation it is assumed that the condition that the movement must be ischoric has been taken into account, and that the stress has been accordingly computed by using a reduced strain description with $\overline{C}$, as outlined earlier in the subsection about the stress field[3].

In addition to the virtual work equation the actual condition is reached in which the movement is isochoric. In a way similar to the methods based on a Hu-Washizu principle, it is required that the following functional is invariant:

$$\Pi_p(\Phi, \overline{J}, p) = \int_{\Omega_0} U(\overline{J})\mathrm{d}\Omega_0 + \int_{\Omega_0} p(J - \overline{J})\mathrm{d}\Omega_0 \tag{5.2}$$

The variable J, the Jacobian, is connected to the solution Φ, hence the dependence of the functional Π_p on Φ. However, the variables $\overline{J}$ and p are independent field variables. The integration is over the reference domain since we assume that the fields $\overline{J}$ and p are body fields (moving along with the coordinate system). In order for the incompressibility condition to be fulfilled, it is required that **the sum of the virtual work equation and the total variation of Π_p disappears**. The function $U(\overline{J})$ may be specified, for example, as

$$U(\overline{J}) = \frac{\kappa_p}{2}(\overline{J} - 1)^2 \quad \text{and thus} \quad U'(\overline{J}) = \kappa_p(\overline{J} - 1)\ ,$$

which can be seen as an artificially introduced volumetric strain energy function, hence the (large) penalty coefficient κ_p can be seen as a bulk compressibility coefficient. This function should be zero, and at the same time the independent field variable $\overline{J}$ should be approximately equal to the Jacobian J, which is related to the actual solution Φ. For this reason a Lagrange parameter field p exists in the functional. Formally, the requirement is:

$$\delta W(\Phi, \delta\mathbf{v}) + \delta\Pi_p(\Phi, \overline{J}, p) = 0 \tag{5.3}$$

Hereby the variations are taken in the directions $\delta\mathbf{v}$, corresponding to the solution Φ; $\delta\overline{J}$ corresponding to the field $\overline{J}$, and δp corresponding to the field p. The whole purpose of this

[3]Strickly speaking, the reduced kinematic description should also be applied for the computation of the viscous stress component, using a reduced deformation rate tensor $\overline{D}$.

exercise, as will be seen below, is to obtain a discretization that makes it possible to enforce the incompressibility condition in an approximate way without locking problems. We have the following partial directional derivatives of Π_p:

$$\mathsf{D}\Pi_p[\delta \mathbf{v}] = \int_{\Omega_0} p\mathsf{D}J[\delta \mathbf{v}]\mathrm{d}\Omega_0 \tag{5.4}$$

$$\mathsf{D}\Pi_p[\delta \bar{J}] = \int_{\Omega_0} (\frac{dU(\bar{J})}{d\bar{J}} - p)\delta \bar{J}\mathrm{d}\Omega_0 \tag{5.5}$$

$$\mathsf{D}\Pi_p[\delta p] = \int_{\Omega_0} (J - \bar{J})\delta p\mathrm{d}\Omega_0 \tag{5.6}$$

The first of these directional derivatives can be reformulated as follows, see (A.4):

$$\mathsf{D}\Pi_p[\delta \mathbf{v}] = \int_{\Omega_0} pJ\mathrm{div}\,(\delta \mathbf{v})\mathrm{d}\Omega_0 = \int_{\Omega_0} pJI : \nabla\delta \mathbf{v}\mathrm{d}\Omega_0 \quad = \int_{\Omega_t} pI : \nabla\delta \mathbf{v}\mathrm{d}\Omega_t$$

The determinant J falls out since the last intergral is over the current configuration, to make it compatible with the way the virtual work equation was written. Since the directional derivatives are added to the virtual work equation via (5.3) it becomes meaningful to understand pI as an artificial pressure term of the stress tensor: It is added to maintain the incompressibility condition.

It is necessary to find an appropriate way to approximate the incompressibility condition in a discretization method without locking. Obviously, it cannot be fulfilled point-wise because, simply speaking, there would be more requirements on the movement than there are coefficients (node variables) to describe it. So it will be assumed that while shape functions N_a are used for the description of the kinematic variables such as velocity and displacements, different shape functions P_a are used for the discretization of the pressure field p and of the field $\bar{J}$, and of course for their variations[4]. It is prudent to use a lower polynomial order for the pressure approximation than for the approximation of the movement.

The second of the above derivatives must be zero, since it is the only contribution to (5.3) depending on the variation $\delta\bar{J}$ and since (5.3) must hold for arbitrary variations of $\bar{J}$. Using the discretization with the shape functions P_a, one obtains:

$$\int_{volo} \left(\sum_a U'(J)P_a\delta\bar{J}_a - \sum_a\sum_b P_bP_a\delta\bar{J}_a p_b \right) \mathrm{d}\Omega_0 = 0 \tag{5.7}$$

Since the variations $\delta\bar{J}_a$ are arbitrary,the following linear equation is obtained. This must be solved for the pressure coefficients p_b:

$$\sum_b \left(\int_{\Omega_0} P_aP_b\mathrm{d}\Omega_0 \right) p_b = \int_{\Omega_0} U'(J)P_a\mathrm{d}\Omega_0 \tag{5.8}$$

This method can be seen as a generalization of the "mean dilational technique" described in the book by Bonet and Wood ([**3**]), which would be obtained if the pressure is assumed constant over the domain of integration, that is, only one shape function $P_a = 1$ would be used.

However, it should not be forgotten that there is one other constraint on the movement, namely the impenetrability condition. As shown in the subsection on collision, this may be

[4]Note that J is not interpolated the same way since it depends on the displacement field which is described by different means.

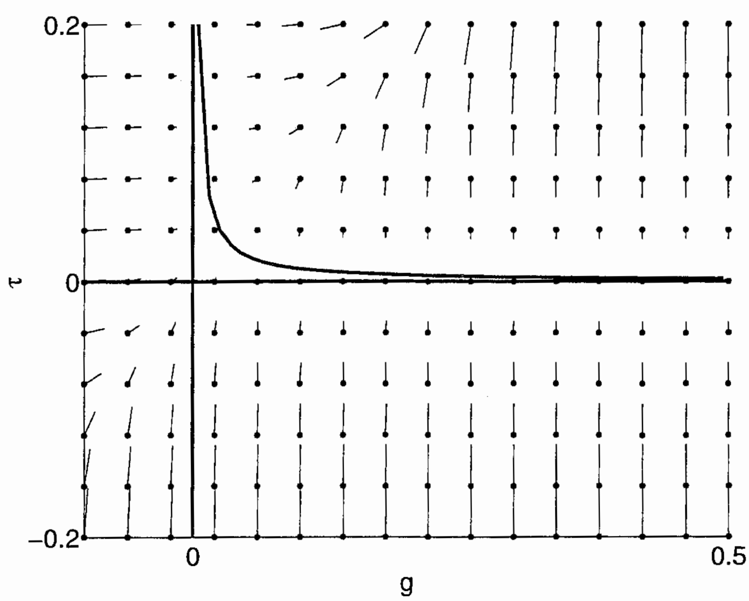


FIGURE 5. *The negative gradients of the function $w(g, \tau)^2$ point towards the constraint $\tau g - \epsilon$ everywhere.*

achieved via the additional requirement that a function $w(g(\Phi), \tau)$ on the surface disappears. This function is the same as was described earlier to approximate the exact collision constraint. The intention is now to make this requirement only point-wise, and in a quadratice sense. The surface that may be in contact is sampled at several points where the condition $w(g, \tau)^2 = 0$ is applied to obtain the traction τ in the direction of a unit vector $\mathbf{m}$ so that the impenetrability condition is enforced. At the contact surfaces the traction force $\tau\mathbf{m}$ must be in equilibrium with the stress field of the body. Therefore, for all contact surface points the condition is:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \tau \mathbf{m} \quad .$$

For this reason, the traction $\tau\mathbf{m}$ becomes part of the surface tractions $\mathbf{t}$.

The above would also be obtained if it had been stipulated that the constraint $w(g, \tau) = 0$ and not its square these two are obviously equivalent. However, if the constraint is to be linearized, it would be benificial to use the quadratic constraint. This is illustrated by the plot in figure Fig. 5. Everywhere in the plane the negative gradient points towards the line where the constraint is met. This is, by the way, not true for the gradients of the square of the constraint $\tau g - \epsilon = 0$.

6. Discretized equations of motion

In consideration of the derivations for the mass matrix, (see 4.5), the lumped stess (see 4.13), the body forces (see 4.7), and the surface traction forces (4.14) and (4.19), the following system can be put together:

$$M\ddot{\underline{x}} + F^{\sigma}(\underline{x}, \dot{\underline{x}}) + H(\underline{x})\underline{p} = F^{\text{int}}(\underline{x}, T) + F^{\partial\Omega_t}(\underline{x}) + F^{\text{col}}(\underline{x}) \tag{6.1}$$

$$Q(\underline{x})\underline{p} = \underline{q}(\underline{x}) \tag{6.2}$$

$$w(g(\underline{x}), \tau(\underline{x}))^2 = 0 \tag{6.3}$$

In these equations the node locations are arranged as one column vector $\underline{x}$, and the pressure coefficients as $\underline{p}$. The dependency on node locations and velocities is given in the terms in the equation. The coefficient T_{aryt} represents the action of the arytenoid (see section 4.2). The inner stress term F^σ further depends on time-variant parameters such as muscle activation of the vocalis muscle, but this dependency is only implied. The matrix H is obtained as follows. The pressure term in the virtual work equation is $\int_{\Omega_t} pI : \nabla\delta\mathbf{v}\mathrm{d}\Omega_t$. The discretization of p with the interpolation functions P_a and of $\delta\mathbf{v}$ with N_a results in:

$$\int_{\Omega_t} pI : \nabla\delta\mathbf{v}\mathrm{d}\Omega_t = \sum_a\sum_b \int_{\Omega_t} (P_b \nabla N_a \mathrm{d}\Omega_t) p_b \delta\mathbf{v}_a = \sum_a\sum_b H_{ab} p_b \delta\mathbf{v}_a \quad . \tag{6.4}$$

The second equation (6.2) for the pressure coefficients p_a in $\underline{p}$ is a short notation for (5.8).

The above system of equations lacks a detailed description of the origin of the surface forces F_t. It is known that they result from the pressure distribution that is delivered by the particular flow pattern near the surface of the vocal folds. It is also clear that this flow pattern and hence the pressure field depends on the state of the vocal folds. From this it becomes obvious that the above system needs to be combined with one more complete dynamic system: the air flow model. The mutual coupling between the two systems consists of the following: The vocal fold model's state provides a boundary condition for the fluid mechanical system, and the air flow provides a pressure field on the surface of the vocal folds.

7. Linearization

The non-linear system of equations describing the discretized movements of the vocal folds can be solved, in general, by two methods. One method consists in eliminating at each time step the pressure variables and surface tractions and proceeding with an explicite method such as the Euler stepping method, or 2nd or higher order Runge-Kutta methods. The advantage of these methods is that no further linearizations are necessary. Only one linear system needs to be solved at any time step to resolve the first equation for $\ddot{\underline{x}}$.

Another method among the explicite solution methods is the central difference method in which the current acceleration and velocity of the dynamic system is approximated in terms of the central differences around the previous time step. This method was applied by Dang and Honda [**5**] for the modeling of a dynamic vocal tract model. This method is staight-forward since their model is directly formulated as an ordinary quasi-linear second order differential equation. A stiffness matrix and a damping matrix can be directly obtained without linearization since the relation between nodal distances and force is assumed to be linear (within a range).

The major disadvantage of explicite methods is that very small time steps may be necessary (especially, of course, during collision) to obtain stable solutions. Implicite methods, on the other hand, while not free from this restriction, can be shown to be more stable but certainly only within limits. In general terms, an implicite method is one in which the current configuration of the dynamic system is used to calculate stresses and forces. All implicite methods contain therefore an iterative step during which a new dynamic equilibrium is computed, unless the system is linear.

While implicite methods are considered more accurate and stable (*c.f.* the discussion in Bathe [**2**], chap. 9), they have the disadvantage that a linearization of the equations of motion at each time step is required. One of the most celebrated version among the implicite time stepping methods is the Newmark method, for this reason the procedure is documented in the appendix of this report (see Appendix F).

In way of linearizing the system of equations, the main difficulty lies in computing a tangential stress-strain relationship that is then lumped into a tangential stiffness matrix. The methods to obtain the tangential stiffness matrix are described in Bathe's book [**2**] and in Bonet's and Wood's book (see [**3**], chapter 7.4).

The central problem in obtaining the tangential stiffness matrix is to obtain a tangential stress-strain relationship based on the constitutive equations for the tissue. Therefore, it is described in this report. For the elastic stress component which can be represented by a hyperelastic term, the second derivative of the elastic potential Ψ is needed. This is the Lagrangian tangential elasticity tensor:

$$\mathfrak{D} = \frac{\partial^2 \Psi}{\partial E \partial E} = 4 \frac{\partial^2 \Psi}{\partial C \partial C} \tag{7.1}$$

Using the reduced potential $\Psi(\overline{C})$, the chain rule is applied to obtain $\mathfrak{D}$. This will be done by first computing an intermediate $\overline{\mathfrak{D}}$ and transforming it to $\mathfrak{D}$. In the sequel the Lagrangian elasticity tensor is transformed into a spatial elasticity tensor.

The transformation from $\overline{S}$ to S (see (4.11)), is achieved by multiplication with a 4th order symbol, that will be denoted $\mathfrak{X}$. In components, this symbol is defined as follows:

$$\mathfrak{X}_{ijkl} = \frac{\partial \overline{C}_{ij}}{\partial C_{kl}} = J^{-2/3}(\delta_{jk}\delta_{jl} - \frac{1}{3}C_{ij}C^{-1}_{kl})$$

The operation of $\mathfrak{X}$ becomes clear by writing out the differentiation of the elastic potential in full:

$$S_{kl} = 2\frac{\partial \Psi(\overline{C})}{\partial C_{kl}} = 2\frac{\partial \Psi(\overline{C})}{\partial \overline{C}_{ij}}\frac{\partial \overline{C}_{ij}}{\partial C_{kl}} = \overline{S}_{ij}\mathfrak{X}_{ijkl}$$

In tensor notation it is possible write the transformation as double tensor contraction:

$$S = \overline{S} : \mathfrak{X}$$

To proceed, the chain rule can then be employed:

$$\frac{\partial S}{\partial C} = \mathfrak{X} : \frac{\partial \overline{S}}{\partial \overline{C}} : \mathfrak{X} + \overline{S} : \frac{\partial \mathfrak{X}}{\partial C}$$

The tedious algebra of this is documented in Appendix D. As a result the following Lagrangian elasticity tensor was found.

$$\begin{aligned} 2\frac{\partial S}{\partial C} = J^{-4/3}&\left[\overline{\mathfrak{D}} - \frac{1}{3}\left((\overline{\mathfrak{D}} : C) \otimes C^{-1} + C^{-1} \otimes (C : \overline{\mathfrak{D}})\right) + \frac{1}{9}(C : \overline{\mathfrak{D}} : C)C^{-1} \otimes C^{-1}\right] \\ &-\frac{2}{3}J^{-2/3}\left[\overline{S} \otimes C^{-1} + C^{-1} \otimes \overline{S} - (\overline{S} : C)(\mathfrak{I} - \frac{1}{3}C^{-1} \otimes C^{-1})\right] \quad , \end{aligned} \tag{7.2}$$

whereby $\mathfrak{I}$ is in components:

$$\mathfrak{I}_{ijkl} = C^{-1}_{ik}C^{-1}_{jl} \, .$$

7.0.1. *Fiber strain example.* Following is an example that will be useful for the vocal fold model but also for the modeling of contractile tissue such as muscles.

The strain energy function may have a component that only depends on the elongation in a fiber direction. The direction of the fiber is given by some unit vector f in the reference configuration. Using the reduced Cauchy tensor, the stretch of the fiber is:

$$l(\overline{C}) = (f^T \overline{F}^T \overline{F} f)^{1/2} = (f_i C_{ij} f_j)^{1/2}$$

Resulting is an energy function that depends entirely on the stretch in the fiber direction, namely $\Psi(\overline{C}) = h(l(\overline{C}))$. The differential of l with respect to $\overline{C}$ is:

$$\frac{\partial l(\overline{C})}{\partial \overline{C}} = \frac{1}{2l(\overline{C})} f \otimes f \quad .$$

The reduced 2nd Piola tensor is found as:

$$\overline{S} = 2\frac{\partial h(l(\overline{C}))}{\partial \overline{C}} = \frac{h'}{l} f \otimes f \quad , \tag{7.3}$$

and the reduced elasticity matrix:

$$\overline{\mathfrak{D}} = 2\frac{\partial \overline{S}}{\partial \overline{C}} = \left(\frac{h''}{l^2} - \frac{h'}{l^3}\right) f \otimes f \otimes f \otimes f \quad . \tag{7.4}$$

It is certainly instructive to take an example of linear elasticity in which the strain energy function is designed to be $h(l) = (a/2)(l^2 - 1)$, where a is some constant. The development above then shows that the Piola tensor is a constant $\overline{S} = af \otimes f$ and the tangential matrix $\overline{\mathfrak{D}}$ disappears.

If the Piola tensor is constant, it doesn't mean that the actual stress, represented by the Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1}FSF^T$, is constant. This depends on the deformation gradient F.

7.0.2. *Transformation to spatial tangent elasticity tensor.* Just as the 2nd Piola tensor is transformed to the Cauchy stress, it is necessary to transform its tangential derivative to a corresponding spatial tangential derivative. It is easy to see that $J^{-1}FC^{-1}F^T = J^{-1}I$, so the following is obtained:

$$\boldsymbol{\sigma} = J^{-1}FSF^T = J^{-5/3}(F\overline{S}F^T - \frac{1}{3}(\overline{S} : C)I) \tag{7.5}$$

The relation between the 2nd derivative of the 2nd Piola tensor and the Eulerian (spatial) tangential elasticity tensor is:

$$\mathfrak{c}_{ijkl} = J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}\mathfrak{D}_{IJKL}$$

It is straight-forward to define an intermediate reduced Eulerian tangential elastic tensor $\overline{\mathfrak{c}}$ by replacing $\mathfrak{D}$ by $\overline{\mathfrak{D}}$ in this transformation. If $\overline{\mathfrak{c}}$ is then transformed to the corresponding unreduced elastic tensor, using the above Piola-transformation, the following formula is obtained:

$$\mathfrak{c} = J^{-7/3}\left[\overline{\mathfrak{c}} - \frac{1}{3}\left((\overline{\mathfrak{c}} : I) \otimes I + I \otimes (I : \overline{\mathfrak{c}})\right) + \frac{1}{9}(I : \overline{\mathfrak{c}} : I)I \otimes I\right] \tag{7.6}$$

So in short, the method to calculate $\mathfrak{c}$, the Eulerian elasticity tensor, comes down to first putting together a material elasticity tensor $\overline{\mathfrak{D}}$, then transforming it via the Piola transformation described above to obtain $\overline{\mathfrak{c}}$.

7.1. Linearization of the viscous stress component.

The viscous stress contribution to the virtual work using the test velocity field δv is:

$$\delta W_v = \int_{\Omega_t} \boldsymbol{\sigma}_v : \delta D \,\mathrm{d}\Omega_t = \int_{\Omega_t} (2\mu D - \frac{2\mu}{3}\mathrm{div}\,(v)\, I) \, : D\mathrm{d}\Omega_t \quad .$$

This can be rewritten as:

$$\delta W_v = \int_{\Omega_t} 2\mu D : \delta D - \frac{2\mu}{3}\mathrm{div}\,(v)\,\mathrm{div}\,(\delta v) \quad .$$

The directional derivative of this functional in the direction of a second field w, keeping δv constant is:

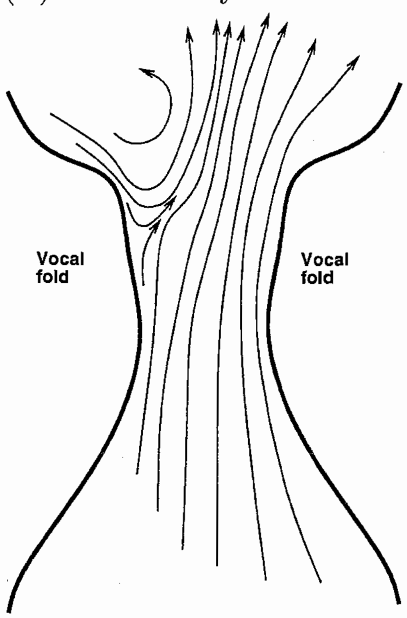
$$\mathsf{D}\delta W_v[w] = \int_{\Omega_t} \left(2\mu D_w : \delta D - \frac{2}{3}\mu \operatorname{div}(w) \operatorname{div}(\delta v) \right) \mathrm{d}\Omega_t \quad .$$

Hereby D_w is the symmetric part of the gradient of the (velocity-) field w.

8. Modeling air flow in the three-dimensional glottis model

The main purpose of modeling the air flow in this model is to obtain a sufficiently accurate computation of the pressure field acting on the surface of the vocal folds. It is clear that, using current computational methods for fluid mechanics (e.g.,[**8**],[**4**]) and computer resources, the accurate flow modeling in the glottis would be a very demanding computational task. Several considerations about the flow field need to be clarified in order to obtain some reasonable simplifications of the task, mainly resulting in faster computation without sacrificing too much of the validity of the computed air flow.

(A) Is the flow symmetric? Or do we regularly have flow regimes as in this speculative sketch?



In the real vocal tract the flow is certainly not (strictly) symmetric. In spite of that, the proposed vocal fold model makes a symmetry assumption. First it should be noted that if the flow is not symmetric, the movements of the vocal folds cannot be symmetric, because an asymmetric flow will create an asymmetric pressure profile on the folds, that must result in asymmetric movements of the folds even if their material properties are identical[5] Nevertheless, it could be argued that if the flow is at least "reasonably" symmetric, the vocal fold model can be reasonably symmetric. The main advantages of modeling a symmetric vocal fold model with symmetric flow are that (a) the collision modeling is drastically simplified, and (b) the flow modeling is drastically simplifed because it is easier to find a flexible mesh for the air flow simulation, as discussed below.

One strong argument against the assumption of symmetry in the flow is the observation of the *Coanda effect*, namely that in a duct with varying cross section the flow tends to lean towards one or the other wall. On the other hand, one important argument put forward against the Coanda effect is to suggest that there is not enough time during the oscillatory cycle of the glottis to establish such an asymmetric flow field. This idea is also supported by explorative experimental evidence in [**12**]: The authors show a series of Schlieren photographs that are taken during the change of an impulsively starting flow in an experimental (rigid, non-moving) glottis model and point out that a Coanda effect can not be observed in that case. The experiment was done after strong evidence for the Coanda effect could be found in steady flow experiments. In my opinion, in spite of this strong experimental evidence, the non-existence

[5]This very trivial and very sharp observation was brought to my attention in communication with Matthias Heil.

of the Coanda effect for the moving glottis is not necessarily the last word. Therefore, it is more a matter of convenience to assume that the air flow can be modeled symmetrically.

8.1. One-dimensional flow approximation.

In the article by Story and Titze [13] a straight-forward one-dimensional flow model is used. It results in a simple algebraic equation that relates the pressure along the glottis to the cross-sectional area. It is assumed that the flow detaches at the point of minimal constriction so that the pressure acting on the glottis above the minimal constriction is equal to the supra-glottal pressure.

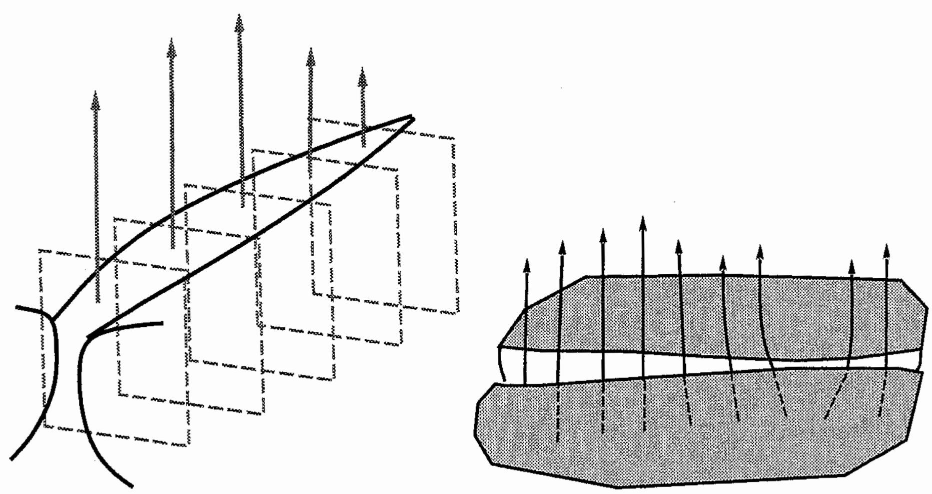
The Bernoulli equation results for the case of uniform flow, described by a volume velocity U, in the following:

$$p = p_s - \frac{1}{2}U^2(\frac{1}{A^2} - \frac{1}{A_s^2})$$

where A_s is the opening area at the beginning of the constriction and A the area along the vocal folds. Further simplifications shown in [13] result in this equation for the pressure:

$$p = p_s - (p_s - p_i)(A_{min}/A)^2 \quad .$$

Hereby, A_{min} is the minimal cross sectional area along the glottis, p_s the sub-glottal, and p_i the supra-glottal pressure. A generalization to two dimensions as flow through multiple parallel channels may be conceivable in which each one is considered a one-dimensional flow:



However, as shown on the right, it would be necessary to set up a communication between two neighboring flows: If the flow gets blocked in one channel, and the neigboring channels are still open, an increase of the flow in the neighboring channels would be expected. This already appears complicated enough to justify (in my view) the investigation of the actual fluid flow problem from first principles[6].

Pelorson *et al* [12] modified the detatchment assumption by investigating more accurately the condition for detatchment of the flow based on a shear-boundary layer approximation. This results in a different definition of the point of detatchment, and also in some improvements of the two-mass model.

[6] In considering generalizations to 2 or 3-dimensional flow, it appears that instead of the algebraic equation that is used to compute the pressure, a Poisson equation is obtained for the pressure in 2 or 3-dimensional flow, as will be shown below.

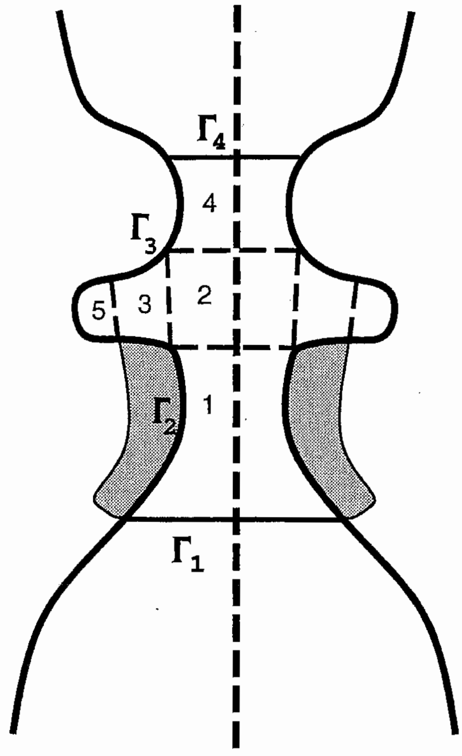


FIGURE 6. *Sketch of various regions for modeling of the fluid dynamics in the symmetric glottis. The regions 1,2 and 3 are moving with the surface of the vocal folds, that is the field* $\mathbf{w}$ *(see text) is defined in these regions depending on the movements of the folds. The regions 4 and 5 are fixed. The region under the boundary* Γ_1 *is the sub-glottal region, and the region* Γ_4 *is the supra-glottal region.*

8.2. Description of flow in a moving grid.

The following gives an outline starting at the general Navier-Stokes equations for fluid flow. The Navier-Stokes equations are formulated in a moving coordinate system, following the method that is known in the literature as **Alternate Lagrangian or Eulerian** (ALE) method. The basic idea is that a grid in which the fluid is simulated moves partially with the moving surface of the deforming body.

In particular (see Fig. 6), between the two (symmetrically moving) vocal folds, the vertical direction of movement is identical to the movements of the vocal folds, that is, the z and y components of the movement is the projection of the movement of the folds onto the y-z plane. The grid's movement in x direction is equal to the movement in x direction on the surface of the vocal folds and equal to zero in the midsagittal plane. Above the glottis, another region's movement is only determined by the movement of the upper edge of the vocal folds, as can be seen in Fig. 6.

Relative to this moving coordinate system, the Navier-Stokes equations can be formulated. Using either a set of polynomial shape functions to model the fluid flow relative to the movements of the grid or by making use of some other method based on the finite volume method or even spectral methods, the flow and pressure fields may be computed. One important problem that has to be solved is the fact that the grid collapses when and where the two folds impact.

8.3. Navier-Stokes equations in a moving coordinate system.

Following the moving grid idea, it is assumed that there is a velocity field $\mathbf{w}$ inscribed on the fluid domain. The velocity $\mathbf{w}$ of the field is in general independent of the spatial velocity field of the fluid $\mathbf{v}$, see Fig. 7. A control volume shown in gray in the figure is moving with the grid, that is it's boundaries are moving at the velocity $\mathbf{w}$ at the spatial locations of the boundary. Transport equations can be stated for any field that is associated with the air flow, see Gurtin's book [**10**], page 78, or Ferzigter and Perić's book [8], chapter 12. In the control volume moving with $\mathbf{w}$, Reynold's transport theorem becomes:

Let Φ be a smooth spatial field and let it be either scalar of vector valued. For any control volume that moves with the field $\mathbf{w}$ the rate of change becomes:

$$\frac{d}{dt}\int_{C_t} \Phi dV = \int_{C_t} \Phi' dV + \int_{\partial C_t} \Phi(\mathbf{v}-\mathbf{w})\cdot \mathbf{n} dA$$

Φ' represents the change of the field, keeping the spatial location fixed, whereas $\dot{\Phi}$ would be the material change of the quantity while moving along with a particle. It should be noted that there are two limiting cases. In the Eulerian description, the field $\mathbf{w}$ is zero and in the Lagrangian description it is equal to the velocity of the movement, so that the second part disappears.

In particular, if the quantity Φ is the momentum $\rho\mathbf{v}$, then the above is valid for each component and we get:

$$\frac{d}{dt}\int_{C_t} \rho\mathbf{v}_j dV = \int_{C_t} \frac{\partial \rho\mathbf{v}_j}{\partial t} dV + \int_{\partial C_t} \rho\mathbf{v}_j(\mathbf{v}-\mathbf{w})\cdot \mathbf{n} dA \quad .$$

In vector notation this amounts to

$$\frac{d}{dt}\int_{C_t} \rho\mathbf{v} dV = \int_{C_t} \frac{\partial \rho\mathbf{v}}{\partial t} dV + \int_{\partial C_t} (\mathbf{v}\otimes\rho(\mathbf{v}-\mathbf{w}))\mathbf{n} dA \quad .$$

The position of the density ρ was deliberately put to the right since the continuum equation can be applied written in a the appropriate form

$$\frac{d}{dt}\int_{C_t} \rho dV = \int_{C_t} \frac{\partial \rho}{\partial t} dV + \int_{\partial C_t} \rho(\mathbf{v}-\mathbf{w})\cdot \mathbf{n} dA = 0$$

The above equation must hold because there are only two sources of how matter can disappear or appear in the control volume C_t: Either because the mass density changes or because matter leaves or enters the volume. This happens at a speed $\mathbf{v}-\mathbf{w}$ relative to the boundary of the control volume, so the second term comprises this transport of mass.

The integral of moment over the control volume must be in equilibrium with the surface forces acting on the surface of the control volume, assuming the absense of any body forces such as gravity. So we get two equations, independent of the control volume (using the divergence theorem).

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}-\rho\mathbf{w})) = \operatorname{div}(\boldsymbol{\sigma})$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}-\rho\mathbf{w}) = 0$$

This is the same as:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \mathbf{v}\,(\rho \mathbf{v} - \rho \mathbf{w}) + \mathbf{v}\mathrm{div}\,(\rho \mathbf{v} - \rho \mathbf{w}) = \mathrm{div}\,(\boldsymbol{\sigma})$$
$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v} - \rho \mathbf{w}) = 0$$

And so the continuity equation can be applied to simplify, by subtracting $\mathbf{v}$ times the second equation from the first one. This results in the Navier Stokes equations($\rho > 0$).

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v}\,(\mathbf{v} - \mathbf{w}) &= \frac{1}{\rho}\,\mathrm{div}\,(\boldsymbol{\sigma}) \\ \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v} - \rho \mathbf{w}) &= 0 \end{aligned} \tag{8.1}$$

If $\mathbf{w} = 0$ the usual Navier-Stokes and continuity equations in the Eulerian description are obtained (just like (3.1) without the f). In the other extreme, if a reference frame is chosen that moves with the deforming body, that is, $\mathbf{w} = \mathbf{v}$, the convective form drops out and the case of a Lagrangian description of the motion is obtained.

Under the assumption of low Mach numbers for the flow field, the air is treated as a Newtonean fluid[7], see [**10**]. The stress field $\boldsymbol{\sigma}$ is then

$$\boldsymbol{\sigma} = -(\pi + \frac{2}{3}\mu \mathrm{div}\,\mathbf{v})I + 2\mu D \tag{8.2}$$

The variable π referes to the pressure that would be there without a movement. The divergence term is designed to remove the volumetric part of the stress due to $2\mu D$ alone, so that the actual pressure is

$$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} = \pi \tag{8.3}$$

It is often preferable to consider the case of incompressible movement to get rid of sound waves. In this case, the diverence of the velocity field must be zero and the pressure can not be related to a density change of the air. The pressure then is arbitrary up to a constant and only its gradient will be used to enforce the incompressibility condition $\mathrm{div}\,\mathbf{v} = 0$. Before the incompressible case is treated, the case of slight compressibility will be considered. For small compressions of air, it can be stated that a constitutive relation exists between the air density and the pressure: $p = \pi(\rho)$. As explained in Gurtin's book (see [**10**], section 19), a wave veloctity $\kappa(\rho)$ can be introduced that follows from the constitutive equation, resulting in a relationship between the gradient of pressure and the gradient of mass density:

$$\kappa^2(\rho) = \frac{d\pi(\rho)}{d\rho} \quad (\text{units } [\mathrm{m/s}]^2) \quad \text{and} \quad \frac{1}{\rho}\nabla p = \frac{\kappa^2(\rho)}{\rho}\nabla \rho \quad .$$

We are dealing with low enough speeds and small enough pressure perturbations (compared to the atmospheric pressure p_0) so it can be assumed that the mass density does not change significantly. Hence the approximation that the mass density is a big constant plus a small fluctuation, and the speed of sound is a constant:

$$\rho = \rho_0 + \tilde{\rho} \quad \text{and} \quad \frac{\kappa^2(\rho)}{\rho} \approx \frac{c_0^2}{\rho_0} \quad \text{and} \quad \nabla \rho = \nabla \tilde{\rho} \quad .$$

[7] It should be noted that for $D = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ the velocity gradient of $\mathbf{v}$ is used and not that of $\mathbf{v} - \mathbf{w}$. This is the case since the viscous stress is only a function of the relative movement of particles in the fluid flow, therefore it is independent of the movement of the reference field whose velocity is $\mathbf{w}$.

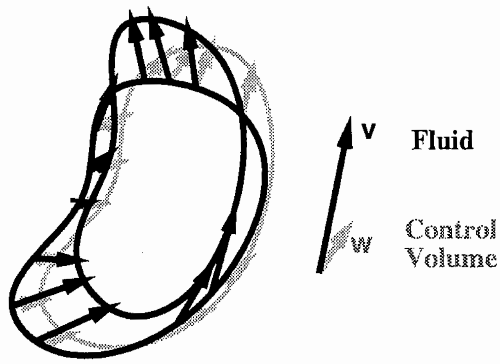


FIGURE 7. *The description of fluid flow in a moving coordinate system. Both the velocity* $\mathbf{v}$ *of the fluid and the velocity* $\mathbf{w}$ *of the grid are relative to the fixed space. The control volume moves with the veloctу field of the grid - its position after a short time is shown in gray. The fluid that initially covered the control volume is soon in the region shown with black lines. The arrows show the displacements of individual particles on the surface of the control volume.*

Using these approximations in the equations of movement and the continuity equation (8.1), together with the linear viscous stress law (8.2), written in the expanded form, the following is obtained:

$$\begin{aligned} \mathbf{v}' + \nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{w}) + \frac{c_0^2}{\rho_0}\nabla\tilde{\rho} &= 2\nu\mathrm{div}\left(D - \frac{2}{3}\nu(\mathrm{div}\,\mathbf{v})\,I\right) = \nu\Delta\mathbf{v} - \frac{1}{3}\nu\nabla\,(\mathrm{div}\,\mathbf{v}) \\ \tilde{\rho}' + (\rho_0+\tilde{\rho})\mathrm{div}\,(\mathbf{v}-\mathbf{w}) + \nabla\tilde{\rho}\cdot(\mathbf{v}-\mathbf{w}) &= 0 \end{aligned} \tag{8.4}$$

The small signal approximation implies a constant linear relation between the gradient of the pressure and the gradient of the density, that is:

$$\frac{1}{\rho_0}\nabla p = \frac{c_0^2}{\rho_0}\nabla\tilde{\rho} \quad \text{and thus} \quad p = c_0^2\tilde{\rho} + const \tag{8.5}$$

The *const* is spatially constant and can only depend on time, so it corresponds to the average atmospheric pressure and will be assumed constant over time as well. The above equations can be rewritten after the divergence term of the second one has been collapsed into one expression again:

$$\begin{aligned} \mathbf{v}' + \nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{w}) + \frac{1}{\rho_0}\nabla p = \nu\Delta\mathbf{v} - \frac{1}{3}\nu\nabla\,(\mathrm{div}\,\mathbf{v}) \\ p' + \mathrm{div}\,[(\rho_0 c_0^2 + p)(\mathbf{v}-\mathbf{w})] = 0 \quad . \end{aligned} \tag{8.6}$$

The pressure p in these equations must be understood as a small fluctuation relative to the atmospheric pressure. The above equation system must be equiped with consistent boundary conditions. At the entrance it is necessary to specify the subglottal pressure p_s:

$$\tilde{p}|_{\Gamma_1} = p_s \quad .$$

If the viscosity stress term is kept, the velocity field must follow the "non-slip" condition, that is, the velocity $\mathbf{v}$ must be equal to the movement of the grid, $\mathbf{w}$, at the boundary Γ_2 (see Fig. 6). The pressure at the same boundary appears to the vocal fold as an external pressure.

It is apparant that the above system requires very tiny time steps and high spatial resolution for its solution, since the solution algorithm must resolve the actual sound propagation and at the same time should model with sufficient accuracy the high velocity gradients that can be expected near the walls (shear boundary layer).

The following approximations can be made. First, a small reformulation of the system for the case of neglecting viscosity ($\nu = 0$) should allow significantly lower spatial resolution. The boundary conditions must be changed to a slip condition which means that only the velocity term perpendicular to the boundary disappears, since now the particles are no longer assumed to stick to the boundary. The reasoning for this assumption is that for the air flow far from the walls the viscous term is almost irrelevant because small velocity gradients will be found. It is then convenient to assume that there will be a thin boundary layer where the viscosity is relevant. So the slip condition implies the assumption that this layer is very small. See also the discussion of shear layer boundary in the article by Pelorson *et al* [**12**].

An acoustic approximation can be obtained by taking the time derivative of the pressure equation and the divergence of the moment equation. Then the moment equation is multiplied with $\rho_0 c_0^2$ and the term $\operatorname{div}\mathbf{v}$ eliminated. This results in the following.

$$p'' - c_0^2\Delta p + \frac{d}{dt}\operatorname{div}\left[p(\mathbf{v}-\mathbf{w})\right] = \rho_o c_0^2\left(\operatorname{div}\mathbf{w}' + \operatorname{div}\left[\nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{w})\right]\right) \tag{8.7}$$

This equation is still exact within the small pressure fluctuation approximation, since nothing else was neglected. In fact, what is seen here is a wave equation (the term $p'' - c_0^2\Delta p$) with a rather complicated looking source term. If the density field (resulting from the continuity equation) had not been replaced by a small fluctuation pressure field, the resulting of the operations above would be the equations attributed to Lighthill, and derived in Dowling and Williams [**6**], section 7.4. There is shown how the Navier-Stokes equations (momentum and continuity) can be formulated as non-linear wave equations.

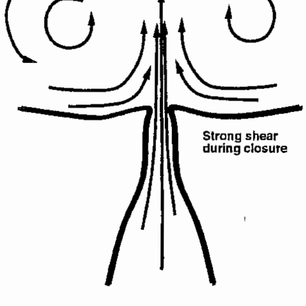
The above is interesting by itself but does not yield any simplifications, unless it is now argued that the spreading of the pressure wave is so fast, compared to the size of the glottis, that the pressure signal occurs instantaneously everywhere. So the case of infinite signal speed of an incompressible gas would be the limiting case. To get to such an approximation, the next idea is to maintain only terms that contain c_0^2, and then assume that the velocity field is solenoid ($\operatorname{div}\mathbf{v} = 0$). This results in:

$$\Delta p = -\rho_0\operatorname{div}\mathbf{w}' - \rho_0(\nabla\mathbf{v}) : (\nabla\mathbf{v} - \nabla\mathbf{w})^T \tag{8.8}$$

Another approach can be found in applying the incompressible assumption from the beginning. This leads to a very similar Poisson equation for the pressure (see below) that does not include the term from the moving mesh (it should be there).

8.4. Potential flow.

For the glottis, the assumption is often made that there are no significant eddies as long as the flow converges. There may be eddies above the glottis, and it can be assumed that they shear off the edge of the rapidly closing vocal folds, as shown in the following illustration:



For most of the flow field between the vocal folds, not above them, the assumption of no eddies may be quite reasonable. It is certainly conventient. If it is thus assumed that the rotation of the velocity field $\mathbf{v}$ disappears, this is equivalent to the assumption that the velocity gradient is symmetric. Further, under this assumption the velocity field can be written as the gradient of a velocity potential ϕ. The moment equation in this case can be reformulated as follows.

$$\nabla\left(\phi' + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho_0}\right) = \nabla(\nabla\phi)\cdot\mathbf{w} \quad , \tag{8.9}$$

or in the following way (see Appendix E.1):

$$\nabla\left(\phi' + \frac{1}{2}[(\nabla\phi)^2 - \mathbf{w}^2] + \frac{p}{\rho_0}\right) = (\nabla\mathbf{w})^T(\nabla\phi - \mathbf{w}) \tag{8.10}$$

In any case, a modified Bernoulli equation is not obtained since for that the right hand side term should disappear (see Gurtin's book for details). This could be achieved only if the movements of the glottis that enter the equations by means of the moving grid velocity $\mathbf{w}$ are neglected. It could also be argued that a Bernoulli equation can not be obtained due to the fact that the rotation of $\mathbf{v} - \mathbf{w}$ does not disappear even if $\mathbf{v}$ is an irrotational flow.

However, if it could be assumed that the flow is stationary, then not only would the term ϕ' disappear but also the movements of the grid would be neglected in the equations. See also McGowan's interesting review letter for a discussion of the validity of the assumption of stationarity [**11**].

Remark: If stationarity and potential flow is a reasonable assumption, there is no real need to model the air flow by using the ALE method over the changing domain between the folds. Instead, the flow can be obtained by solving a Laplace equation for the flow potential. Subsequently, the pressure can be obtained from the Bernoulli equation. It is not even necessary to have a grid at all in that case. The Laplace equation for the flow potential can be solved numerically with the boundary element method (see Banerjee, [**1**]). This needs to be further investigated. Problems can be expected especially during the collision of the glottis - of course.

8.5. The incompressible case and another Poisson equation for the pressure.

It is often assumed that the flow for low Mach numbers (as is the case in the glottis) is appropriately modeled under the assumption that the air is incompressible. In the above method this is considered as a limiting case: The signal speed c_0 is infinite in the limit, and the divergence of the velocity field $\mathbf{v}$ disappears. Essentially, c_0^2 plays the role of a penalty coefficient.

In the following approach the incompressibility condition is used *apriori*, not allowing a small signal approximation for the pressure field. In this case the Navier Stokes equations become:

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\mathbf{v}\,(\mathbf{v} - \mathbf{w}) + \frac{1}{\rho_0}\nabla p = 2\nu\,\mathrm{div}\,D \quad ,$$
$$\mathrm{div}\,\mathbf{v} = 0 \quad .$$

For incompressible flow, the pressure field plays the role of a Lagrange multiplier variable. It must be such that the incompressibility condition is valid. Just like in solid mechanics - there is no direct connection between the pressure field and the flow field. Indirectly however, if the moment equation is resolved for the gradient of the pressure p and the divergence is taken, a Poisson equation for the pressure is obtained. The incompressibility condition is implicitely contained in this equation, $\mathrm{div}\,\mathbf{v} = 0$ is taken into account everywhere.

In computing the divergence of the moment equations, it can be first stated that div $\frac{\partial \mathbf{v}}{\partial t} = 0$ must be true due to the incompressibility condition. (This is the reason why the Poisson equation can be solved alternating with the solution of the velocity equation, resulting in what is called the velocity correction method). Further, the double divergence of the symmetric deformation rate tensor dissappears if $\operatorname{div}\mathbf{v} = 0$ (see Appendix 3). The only term left is $\operatorname{div}(\nabla\mathbf{v}(\mathbf{v}-\mathbf{w}))$ (see Lemma 4). The following Poisson equation for the pressure field is obtained:

$$\Delta p = -\rho_o \operatorname{tr}(\nabla\mathbf{v}\ \nabla(\mathbf{v}-\mathbf{w})) = -\rho_0(\nabla\mathbf{v}) : (\nabla\mathbf{v} - \nabla\mathbf{w})^T \tag{8.11}$$

It can be seen that the term containing the accelaration of the grid in (8.8) is missing in the above equation. However, the previous Poisson equation for the pressure was obtained by taking the limit ($c_0 \to \infty$) of the acoustic equation for the pressure.

8.6. Description of the grid movement.

In this subsection, a grid is described that represents the "wobbling" space between the vocal folds and the midsagittal plane. It is obtained by interpolating between the movement of the glottal fold's surface points and their projections onto the midsagittal plane. Near the surface of the vocal fold the grid moves along with the movements of the surface, and near the midsagittal plane it only moves along with the y and z component of the movement. The surface of the vocal fold can be parameterized by two parameters (α, β). In particular, for the location and velocity fields at the surface of the fold this becomes:

$$\dot{\mathbf{x}} = \sum_a N_a(\alpha,\beta)\dot{\mathbf{x}}_a \quad \text{and} \quad \mathbf{x} = \sum_a N_a(\alpha,\beta)\mathbf{x}_a \quad ,$$

whereby N_a denotes the shape functions in the vocal fold restricted to the surface. In the model, the z-coordinate is the main flow direction, the y-axis the length axis of the glottis, and the x-axis the horizontal dimension. The distance of a point $\mathbf{x}$ on the vocal fold from the midsagittal plane is therefore $-\mathbf{x}_1$ (since the fold is in the half space of the negative x-axis.). It will be assumed that the grid's movement in the midsagittal plane is the projection of the movement of the fold's surface onto the y-z plane. So one can construct a parameterization of the control volume, based on a parameterization of the surface of the fold (by α and β) and a third parameter γ that is 0 on the glottis, and 1 on the midsagittal plane. This results in the following veloctity field for the grid movements:

$$\mathbf{w} = ((1-\gamma)\dot{\mathbf{x}}_1\,,\ \dot{\mathbf{x}}_2\,,\ \dot{\mathbf{x}}_3)^T$$

Thus, the "wobbling" domain is parameterized by a mapping from a unit cube by three parameters:

$$\mathbf{y} = ((1-\gamma)\mathbf{x}_1\,,\ \mathbf{x}_2\,,\ \mathbf{x}_3)^T \quad ,$$

whereby the dependence of $\mathbf{x}$ on α and β was implied.

The volume element used for integration is not simply $d\alpha d\beta d\gamma$ because the parameterization is based on the parameterizaton of the vocal fold surface. It must be computed by differentiation of the mapping, or equivalently by evaluating the geometry, as follows. The surface normal area element on the fold is (see 4.15):

$$\mathbf{n}dA = \frac{\partial\mathbf{x}}{\partial\alpha} \times \frac{\partial\mathbf{x}}{\partial\beta} d\alpha d\beta \quad .$$

The volume element is

$$dV_m = -\mathbf{x}_1 \left(\frac{\partial \mathbf{x}_2}{\partial \alpha} \frac{\partial \mathbf{x}_3}{\partial \beta} - \frac{\partial \mathbf{x}_3}{\partial \alpha} \frac{\partial \mathbf{x}_2}{\partial \beta} \right) d\alpha d\beta d\gamma \quad .$$

It is also necessary to know the gradient of this field. It is:

$$\frac{\partial \mathbf{y}}{\partial \boldsymbol{\alpha}} = \Lambda = \begin{pmatrix} (1-\gamma)\frac{\partial \mathbf{x}_1}{\partial \alpha} & (1-\gamma)\frac{\partial \mathbf{x}_1}{\partial \beta} & -\mathbf{x}_1 \\ \frac{\partial \mathbf{x}_2}{\partial \alpha} & \frac{\partial \mathbf{x}_2}{\partial \beta} & 0 \\ \frac{\partial \mathbf{x}_3}{\partial \alpha} & \frac{\partial \mathbf{x}_3}{\partial \beta} & 0 \end{pmatrix} \tag{8.12}$$

A gradient operator in spatial coordinates can therefore be formed via

$$\nabla = \Lambda^{-T} \nabla_{\boldsymbol{\alpha}} \quad , \tag{8.13}$$

where the symbol nabla operator with $\boldsymbol{\alpha}$ represents differentiation with respect to the parameters (α, β, γ). The transposed inverse is:

$$\Lambda^{-T} \quad = \quad \frac{1}{x_1} \frac{1}{x_{2,\alpha} x_{3,\beta} - x_{2,\beta} x_{3,\alpha}} \begin{pmatrix} 0 & 0 & n_1 \\ -x_1 x_{3,\beta} & x_1 x_{3,\alpha} & (n_2 + \gamma x_{1,\alpha} x_{3,\beta}) \\ x_1 x_{2,\beta} & -x_1 x_{2,\alpha} & (n_3 - \gamma x_{1,\alpha} x_{2,\beta}) \end{pmatrix} \tag{8.14}$$

It is clear that while the volume element shrinks to zero volume during collision, the transformation matrix becomes singular. It can be hoped though that the singularity can be cancelled out. Integrating over the grid, under an integral the term dV_m is introduced due to transforming the integral back to the parametizing cube. Therefore, the Λ^{-T} usually occures together with dV_m:

$$dV_m \nabla = \begin{pmatrix} 0 & 0 & n_1 \\ -x_1 x_{3,\beta} & x_1 x_{3,\alpha} & (n_2 + \gamma x_{1,\alpha} x_{3,\beta}) \\ x_1 x_{2,\beta} & -x_1 x_{2,\alpha} & (n_3 - \gamma x_{1,\alpha} x_{2,\beta}) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial \beta} \\ \frac{\partial}{\partial \gamma} \end{pmatrix} \tag{8.15}$$

This operator is defined even in the case of the collapsing grid, that is, for $x_1 = 0$.

It is clear from the above that any volume integral, and thus any equations between coefficients describing the velocity field, are inherently and non-linearly related to the location (and velocities) of the nodes describing the movement of the surface of the vocal folds. It should therefore also be noted that any variations in these nodes will affect the field $\mathbf{w}$.

9. Acoustic coupling of the flow simulation

In principle, the flow computation could be stretched over the whole vocal tract. But this seems unnecessary since for most of the vocal tract, the particle velocities are small enough so that the non-linear terms in the Navier Stokes equation can be neglected. Further, it is convenient for lower frequency ranges, to make a one-dimensional acoustic approximation for the region of low flow velocity along the vocal tract. In general, the walls may be moving (either fast in very small amplitudes due to interactions with the sound field, or very slowly due to articulatory movements). For one-dimensional sound propagation in a tube with varying area, $A(x,t)$, the velocity field can be represented by a volume velocity field U which relates to the (one-dimensional) velocity field in the direction of the tube as follows:

$$v(x,t) = U(x,t)/A(x,t) \tag{9.1}$$

Differentiation of this and using the expressions

$$B = ln(A) \quad , \; B_x = A_x/A \quad \text{and } B_t = A_t/A \quad ,$$

whereby differentiation with respect to time and space are marked by the indices, results in the following specializations:

$$\begin{aligned} v_t + \frac{1}{\rho_0} p_x = 0 \quad &\text{becomes} \quad U_t - UB_t + \frac{A}{\rho_0} P_x &= 0 \\ p_t + \rho_0 c_0^2 v_x = 0 \quad &\text{becomes} \quad P_t + \rho_0 c_0^2 (U_x - UB_x) &= 0 \end{aligned}$$

Taking the spatial derivative of the first and the time derivative of the second, most of the terms with the flow variable can be eliminated to obtain:

$$B_x P_x + P_{xx} - \frac{1}{c_0^2}(B_t P_t + P_{tt}) = \frac{\rho_0}{A}(U_x B_t - U_t B_x) \tag{9.2}$$

If it can be maintained that A_x/A^2 and A_t/A^2 are very small, is it possible to separate the pressure and flow fields. Under this assumption the Webster horn equation is obtained:

$$(A_x/A)P_x + P_{xx} = \frac{1}{c_0^2}((A_t/A)P_t + P_{tt}) \tag{9.3}$$

It is assumed that the coupling of the flow field and the acoustic field takes place at some cross-sectional interface surface Γ_i. To connect the pressure fields P and p and the velocity fields U and $\mathbf{v}$, it must be averaged over the interfaces:

$$\begin{aligned} P &= \frac{1}{\text{area}(\Gamma_i)} \int_{\Gamma_i} p dA \\ U &= \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} dA \end{aligned} \tag{9.4}$$

This type of coupling is appropriate for the interfaces Γ_1 to the subglottal space and at Γ_4 to the supra-glottal space (see Fig. 6).

10. Methods for air flow simulation

In the following I will put together the equations that are the starting point of a simulation. Statement of the boundary conditions appears to be a hairy issue. It is assumed that the lungs are an infinite pressure reservoir with a pressure of $p_s(t)$. Supra-glottally (at the down stream end) we apply the condition that the air pressure must be $p_0(t)$. Initially it will be assumed that these pressures are constant. However, in planned extension, the sound field needs to be considered outside of the computational domain for the fluid flow. In that case, the pressure in the vocal tract will interact with the fluid flow via the boundary condition of the pressure at the interface between fluid and acoustic fluid. It can be expected that there will be an effect on the flow velocity field. For instance, a pressure change at the down stream interface will result in a change of the total flow field including the flow at the sub-glottal interface, where the pressure is supposed to be p_s.

Viscous terms will be neglected, so a slip condition will be applied on the surface of the walls, including the moving vocal fold surface.

The boundary conditions for the pressure on these surfaces are obtained by resolving the Navier-Stokes equation for the pressure and multiplying with the surface normal. The following equations are obtained.

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v}(\mathbf{v} - \mathbf{w}) = -\frac{1}{\rho_o}\nabla p \tag{10.1}$$

$$\Delta p = -\rho_o \mathrm{tr}\left(\nabla \mathbf{v}\ \nabla(\mathbf{v} - \mathbf{w})\right) - \rho_o \mathrm{div}\,\mathbf{w}' \tag{10.2}$$

$$p = p_o(t) \quad \text{supra-glottis boundary} \tag{10.3}$$

$$p = p_s(t) \quad \text{sub-glottis boundary} \tag{10.4}$$

$$(\nabla p + \rho_0 \mathbf{a}) \cdot \mathbf{n} = 0 \quad \text{on moving glottis surface} \tag{10.5}$$

$$(\mathbf{v} - \mathbf{a}) \cdot \mathbf{n} = 0 \quad \text{on moving glottis surface} \tag{10.6}$$

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on rigid wall surfaces} \tag{10.7}$$

See Fig. 6 for explanation of the regions. The vector field $\mathbf{a}$ is the instantaneous acceleration of the vocal folds and on the surface of the folds it is equal to $\mathbf{w}'$.

Remark: It should be noted that no boundary conditions are specified for the velocity field on the input and output domain surfaces. Only pressure conditions are specified, following the considerations about the nature of the problem. However, it is usually found in the literature that explicit boundary conditions for the flow field are specified, for example, such as that the flow is perpendicular to the inflow and outflow boundaries. I wonder if this is essential. My reasoning is the following. It is assumed that the equations above are simulated starting with a zero initial velocity field and for simplicity without movement of the vocal folds. So the pressure is computed via a Laplace equation, since the inhomogeneous term in (10.2) is initially zero. The resulting pressure field should be a smooth interpolation between the pressure at the lower and upper ends of the fluid flow domain. Next the flow is updated and a small little bit of flow will occur that influences the pressure field. After that, the pressure field is updated again, this time via the Poisson equation (10.2), and so forth. Essentially, the pressure field provides the potential from which the air flow "is pumped through the domain". I do not see the physical need for specifying the flow boundary conditions for the surfaces Γ_1 and Γ_4 (see Fig. 6), and I would not know how to specify a meaningful boundary condition for the flow. So I leave it unspecified. *This will have to be critically reviewed.*

10.1. Discretization.

In this subsection I will only cover a discretization method via the finite element method or with spectral methods. This may later be specialized to the more common volume element method which can be found partially covered in the book by Ferziger and Perić [**8**]. It should also be possible to use boundary element methods (see Banerjee, [**1**]), and an investigation of that is forthcoming.

Following the Galerkin method the equation for the velocity is multiplied by some arbitrary vector field $\delta\mathbf{v}$ that fulfills the essential boundary conditions, and the resulting inner product field is integrated over the fluid domain. Similarly, for the pressure equation a scalar field δp is used. For the discretization, shape functions Q_a are used to discretize the velocity fields, and shape functions R_a to discretize the pressure fields. In the following the discrete system of equations will be developed without considering singularities. Then, the problems associated with the collapsing grid will be addressed. In the following equations, summation over expressions containing multiple indices is implied.

$$\begin{aligned}
\mathbf{v} &= Q_a \mathbf{v}_a \\
\nabla \mathbf{v} &= \mathbf{v}_a \otimes \nabla Q_a \\
\delta \mathbf{v} &= Q_a \delta \mathbf{v}_a \\
p &= R_a p_a \\
\delta p &= R_a \delta p_a
\end{aligned}$$

In the velocity field, the convective term becomes (switching as needed between components and tensor notation, and using the notation $Q_{a,i} = (\nabla Q_a)_i$):

$$\delta \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v} = Q_a \delta \mathbf{v}_{ai} (\mathbf{v}_b \otimes \nabla Q_b)_{ij} Q_c \mathbf{v}_{cj} = Q_a \delta \mathbf{v}_{ai} \mathbf{v}_{bi} Q_{b,j} Q_c \mathbf{v}_{cj} = Q_a Q_c \nabla Q_b \delta \mathbf{v}_a (\mathbf{v}_b \otimes \mathbf{v}_c)$$

and

$$\delta \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = Q_a \delta \mathbf{v}_{ai} (\mathbf{v}_b \otimes \nabla Q_b)_{ij} \mathbf{w}_j = Q_a \nabla Q_b \delta \mathbf{v}_a (\mathbf{v}_b \otimes \mathbf{w})$$

The pressure gradient in the velocity equation results in a term:

$$\delta \mathbf{v}_a Q_a \frac{1}{\rho_o} R_c p_c$$

After integrating over the domain and dropping the arbitrary $\delta \mathbf{v}$ the equation is obtained:

$$M^Q_{ab} \mathbf{v}'_a + F_a + G_{ac} p_c = 0 \quad , \tag{10.8}$$

whereby

$$\begin{aligned}
M^Q_{ab} &= \int_{V_f} Q_a Q_b dV \quad , \\
F_a &= \int_{V_f} Q_a Q_c \nabla Q_b dV \cdot (\mathbf{v}_b \otimes \mathbf{v}_c) - \int_{V_f} Q_a \nabla Q_b (\mathbf{v}_b \otimes \mathbf{w}) dV \quad \text{and} \\
G_{ac} &= \frac{1}{\rho_o} \int_{V_f} Q_a R_c dV \quad .
\end{aligned}$$

The integration involving $\mathbf{w}$ does not allow (or need) to factor out $\mathbf{w}$, since it is actually only represented via the shape functions used on the surface of the vocal fold model. In the numerical integration, $\mathbf{w}$ must be evaluated at each Gauss point.

The slip condition for the velocity field $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{n} = 0$ can be enforced by an algebraic constraint. It can be obtained from a weak form of the boundary condition that is integrated over the vocal fold surface Γ_2. For this, another scalar Lagrange multiplier field, say η, must be introduced that lives on the surface of the vocal tract. The slip condition then becomes:

$$\int_{\Gamma_2} \eta (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n} dA = 0 \quad \text{for any } \eta \ . \tag{10.9}$$

For the discretization the fact that the field $\mathbf{w}$ must be equal to the velocity of the surface of the vocal folds can be made use of. Conventiently, η can be discretized using the same shape functions as used in the vocal fold model for the displacements, namely N_a restricted to the surface of the vocal folds and the velocities of the surface nodes $\dot{\mathbf{x}}_a$. After integration and dropping the arbitrary coefficients η_a, the following constraint equation is obtained:

$$B^{NQ}_{ab} \mathbf{v}_b = B^{NN}_{ac} \dot{\mathbf{x}}_c \tag{10.10}$$

Hereby, the summation over equal indices is again implied, and the two matrices (which are in general of different order) are:

$$B_{ab}^{NQ} = \int_{\Gamma_2} N_a Q_b \mathbf{n} dA$$

$$B_{ac}^{NN} = \int_{\Gamma_2} N_a N_c \mathbf{n} dA$$

For the pressure equation, use is made of:

$$\delta p \Delta p = \operatorname{div}(\delta p \nabla p) - \nabla \delta p \cdot \nabla p \quad ,$$

and the divergence theorem to incorporate the natural boundary conditions on the surface of the vocal folds, namely by using (10.6):

$$\int_{\Gamma_2} \delta p \nabla p \cdot \mathbf{n} dA = -\int_{\Gamma_2} \delta p \mathbf{a} \cdot \mathbf{n} dA$$

The right hand side term in the pressure Poisson equation becomes, after integrating its product with $\delta p_a R_a$, and dropping δp_a:

$$B_a = \rho_0 \left(\int_{V_f} R_a (\nabla Q_b \otimes \nabla Q_c) dV \right) : (\mathbf{v}_b \otimes \mathbf{v}_c) - \rho_0 \left(\int_{V_f} R_a \nabla \mathbf{w} \cdot \nabla Q_b dV \right) \cdot \mathbf{v}_b$$

(In components the integrant of the second integral is: $R_a Q_{b,j} \mathbf{w}_{i,j} \mathbf{v}_{bi}$.) So the Poisson equation for the pressure becomes, after cancelling signs on both sides:

(10.11)
$$M_{ab}^R p_b + A_a = B_a \quad ,$$

with the right hand side B_a from above and

$$M_{ab}^R = \int_{V_f} \nabla R_a \cdot \nabla R_b dV$$

$$A_a = \int_{\Gamma_2} R_a \mathbf{a} \cdot \mathbf{n} dA$$

10.2. What happens when the grid collapses?

The integrals of the previous section are all taken over the fluid domain. It is clear that the fluid domain will shrink to a very flat region and eventually to nothing where the vocal folds collapse. In what way do these integrals behave when this happens?

The fluid domain is parameterized, as described in sect. 8.6, from the parameters of a unit cube. The functional matrix of the mapping from the cube to the fluid domain was denoted Λ. It becomes singular where the grid collapses. The integrals for the computation of the coefficient matrices for the discrete flow velocity equation (10.8) and for the discrete pressure equation (10.11) are evaluated by integrating over the parameterizing cube. Thereby the differential volume of the cube must be multiplied by $\det \Lambda$. For this reason, in the flow velocity equation, the integrants for M_{ab}^Q and G_{ac} disappear gracefully. The two integrals in the column F_a^Q are also not a problem: There is one spatial gradient (of the shape functions) in each integral. The gradients of the velocity fields become singular because of the transformation of the gradient, c.f. equation (8.13). However, in there, $\det \Lambda \Lambda^{-T}$ the singularity is canceled out.

The problem becomes more severe for the discretized pressure equation, since there are two gradients in the integrals, and there is only one $\det \Lambda$ to cancel out the singularity. To deal with this problem it is proposed to multiply the pressure equation (10.2) with a field $\det \Lambda$

and then proceed. This idea certainly gets rid of the singularity. The justification may be by means of a little practical requirement on the arbitrary field δp that was used to obtain a weak formulation of the pressure equation. It would then be required that if the grid collapses, the weighting function used to obtain a weak solution must be zero.

11. Conclusions

In this report it was attempted design a model of a symmetric vocal fold system with body-flow interaction. The model contains a non-linear tissue description (of the elastic properties) of the vocal folds that can be specialized to various hyperelastic material descriptions. The implementation method can be applied to other materials as well and does not rely on the assumption of hyperelastic tissue models. The mathematical methods underlying the modeling of air flow were described using a moving coordinate system that changes instantaneously with the moving of the vocal folds. The air flow is assumed to be symmetric, for convenience and for the sake of significant simplifications. This is justified by the reasoning that significantly asymmetric flow (Coanda effect) may only occure if the vocal folds are not oscillating.

Several important issues could not be finished or covered in writing by the time of submitting this report. (i) Still missing is a rigorous treatment of the interaction between the fluid flow and the movement equations for the vocal folds. It is planned in futher work to obtain this in the frame work of the virtual work equations with constraint as described in section 5 for the vocal folds alone. (ii) While some information is given on useful numerical methods and how they can be applied here, the implementation of a numerical code and numerical experiments could not be finished by the end of my stay at ATR.

12. Appendix A: Definitions

DEFINITION 1. F is the deformation gradient. It is

$$F_{ij} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{p}_j}$$

where $\mathbf{p}$ is the coordinates of a particle in the reference configuration, and $\mathbf{x}$ is the spatial coordinates of the same particle in the deformed configuration. The determinante of F, the Jacobian, is denoted J.

DEFINITION 2. The right Cauchy tensor is $C = F^T F$, and the left Cauchy tensor is $B = FF^T$

DEFINITION 3. The Euler tensor is $E = \frac{1}{2}(C - I)$.

DEFINITION 4. The rate of the deformation gradient and the spatial velocity gradient relate as follows:

$$L = \nabla(v) = \dot{F}F^{-1}$$

DEFINITION 5. The symmetric part of the velocity gradient is $D = \frac{1}{2}(L + L^T)$, and the asymmetric part is $W = \frac{1}{2}(L - L^T)$

DEFINITION 6. The symmetric part of the velocity gradient and the rate of the Euler tensor relate as follows:

$$F^T DF = \dot{E} \quad \text{or} \quad D = F^{-T}\dot{E}F^{-1}$$

DEFINITION 7. The 2nd Piola tensor and the stress tensor are related by the Piola-transform:

$$\sigma = J^{-1}FSF^T \quad , \quad S = JF^{-1}\sigma F^{-T}$$

DEFINITION 8. The deviatoric part of the Cauchy stress tensor has the pressure removed and its trace is zero:

$$\boldsymbol{\sigma}' = DEV(\boldsymbol{\sigma}) = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} = \boldsymbol{\sigma} - pI \quad \text{where} \quad p = \frac{1}{3}\boldsymbol{\sigma} : I = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} \tag{A.1}$$

Correspondingly, formulating the hydrostatic pressure based on the 2nd Piola tensor:

$$p = \frac{1}{3}J^{-1}S : C \tag{A.2}$$

DEFINITION 9. Isochoric means that $\det C = 1$ and $\frac{d}{dt}\det C = 0$. Since

$$\dot{J} = JC^{-1} : \dot{C}$$

we can also write this as condition for isochoric.

DEFINITION 10. The directional or Gateaux derivative of a function or a functional $f(\Phi)$ in the direction u is defined as follows:

$$\mathsf{D}f(\Phi)[u] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f(\Phi + \epsilon u)$$

In particular, the directional derivative of the deformation tensor F in the direction of a spatial field u is:

$$\mathsf{D}F(\Phi)[u] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{\partial(\Phi + \epsilon u)}{\partial \mathbf{p}} = \frac{\partial u}{\partial \mathbf{p}} = \frac{\partial u}{\partial \mathbf{x}}F\ , \tag{A.3}$$

since $F = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}$.

The directional derivative of J is:

$$\mathsf{D}J(\Phi)[u] = J\mathrm{div}\, u = JI : \nabla u \tag{A.4}$$

13. Appendix B: Shape functions and their gradients.

The parametrization of a volume (vocal folds) in its reference configuration is achieved via scalar shape or interpolation functions $M_a(\xi, \upsilon, \zeta)$ whose arguments are the coordinates of a unit cube or similar simple domains. The initial configuration of the body is described by a set of coefficients $\mathbf{p}_a$, (3-tuples) that can be understood as the reference node points. The material coordinates are thus defined as:

$$\mathbf{p} = \mathbf{p}(\xi, \upsilon, \zeta) = \sum_a M_a(\xi, \upsilon, \zeta)\mathbf{p}_a$$

The current location $\mathbf{x}$ can be expressed either by the same or different shape functions, N_a (whose number can also be different), also defined on the cube domain:

$$\mathbf{x} = \mathbf{x}(\xi, \upsilon, \zeta) = \sum_a N_a(\xi, \upsilon, \zeta)\mathbf{x}_a$$

If the shape functions N_a and M_a are the same then N_a are called isoparametric shape functions. It is useful to understand the current location $\mathbf{x}$ as a vector field that is a function of the material coordinates $\mathbf{p}$.

The material and spatial gradients of the shape functions also need to be computed. The material gradients are obtained by differentiating with respect to $\mathbf{p}$, denoted ∇_0 and the spatial gradients are with respect to $\mathbf{x}$, denoted ∇. We have:

$$\Lambda_{ij} := \frac{\partial \mathbf{p}_i}{\partial \xi_j} = \sum_a \frac{\partial M_a}{\partial \xi_j} \mathbf{p}_{a,j} \tag{A.5}$$

For the material gradient of N_a we get:

$$\frac{\partial N_a}{\partial \mathbf{p}_i} \frac{\partial \mathbf{p}_i}{\partial \xi_j} = \frac{\partial N_a}{\partial \xi_j} \tag{A.6}$$

Thus:

$$\Lambda_{ij}(\nabla_0 N_a)_i = (\nabla_\xi N_a)_j$$

So:

$$\nabla_0 N_a = \Lambda^{-T} \nabla_\xi Na$$

The coeffients of Λ^{-T} and $\nabla_0 N_a$ can be precomputed and stored at the Gauss points. They only depend on the reference configuration which is fixed during the computation. The deformation gradient is the differential of the spatial field $\mathbf{x}$ with respect to its material coordinates:

$$F_{ij} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{p}_j} = \sum_a \mathbf{x}_{ai} \frac{\partial N_a}{\partial \mathbf{p}_j}$$

This can be written in tensor notation as:

$$F = \sum_a \mathbf{x}_a \otimes \nabla_0 N_a \quad . \tag{A.7}$$

Finally, the computation of the spatial gradients of the shape functions is necessary:

$$(\nabla N_a)_i = \frac{\partial N_a}{\partial \mathbf{x}_i}$$

This can be found from:

$$\frac{\partial N_a}{\partial \mathbf{x}_i} \frac{\partial \mathbf{x}_i}{\partial \mathbf{p}_j} = \frac{\partial N_a}{\partial \mathbf{p}_j} \iff \frac{\partial N_a}{\partial \mathbf{x}_i} F_{ij} = \frac{\partial N_a}{\partial \mathbf{p}_j} \iff F^T \nabla N_a = \nabla_0 N_a$$

Thus:

$$\nabla N_a = F^{-T} \nabla_0 N_a \tag{A.8}$$

It is clear than that the material or spatial derivative of any material or spatial field that is discretized using the shape functions N_a can be obtained via the gradients of the shape functions. In particular, if we approximate the field $\delta \mathbf{v}$ by $\sum_a N_a \delta \mathbf{v}_a$, then its gradient becomes:

$$\nabla \delta \mathbf{v} = \sum_a \delta \mathbf{v}_a \otimes \nabla N_a \tag{A.9}$$

14. Appendix C: Some Lemmata and notes

Let S be some mix of hyperelastic behavior plus a stress term that results from dissipative terms and is a function of the strain rate. That is:

$$S = 2\frac{\partial \Psi}{\partial C} + L[D]$$

Then two equations for the case of isochoric movements are obtained:

$$(S - 2\frac{\partial \Psi}{\partial C} - L[D]) : \dot{C} = 0$$

$$JC^{-1} : \dot{C} = 0$$

This can only be true if

$$S = 2\frac{\partial \Psi}{\partial C} + L[D] + \gamma J C^{-1} \tag{C.1}$$

Inserting this into the equation for the pressure (A.2) yields:

$$p = \frac{1}{3} J^{-1}(2\frac{\partial \Psi(C)}{\partial C} + L[D]) : C + \gamma \tag{C.2}$$

So the pressure is only defined up to an arbitrary constant, no matter how S was obtained.

The above argument was carried a little further in the book by Bonet and Wood, [**3**], page 127ff. It is there shown for the case of only elasticity the pressure and γ coincide if $\frac{\partial \Psi}{\partial C} : C = 0$. It is then shown that this is equivalent to the argument that the function Ψ must be homogeneous of order 0, *i.e.*, $\Psi(\alpha C) = \Psi(C)$, for arbitray α. Thus, choosing $\alpha = J^{-2/3}$ does not change the strain energy function.

This argument can also be applied to the case in which the stress tensor obtained as gradient of an elastic potential is augmented by a frictional term $L[D]$. It is necessary to make sure that the frictional term or any other additional terms do not contribute to the pressure. This can be achieved by calculating their pressure terms and aniliating them. For example, in the case of $L[D] = 2\mu D$ in the Cauchy stress, the term $\frac{2}{3}\mu \operatorname{div} v I$ is simply removed from the Cauchy tensor.

In practice it will be necessary to find a field γ such that the isochoric movement condition is approxmiately fulfilled. After that the pressure is computed by taking the inner product with C but the field γ must then be subtracted to obtain the true pressure.

LEMMA 1. *Invariance of stress power: The inner product of $\boldsymbol{\sigma}$ and the velocity gradient is the same as the inner product with D due to the symmetry of $\boldsymbol{\sigma}$. This inner product is the stress power. When using the pushed back versions of the tensors, and integrating over the reference configuration, the same stress power integral is obtained. So it holds that:*

For any volume Ω_0 that is deformed into Ω_t:

$$\int_{\Omega_t} \boldsymbol{\sigma} : D d\Omega_t = \int_{\Omega_0} S : \dot{E} d\Omega_0$$

Proof:

$$\begin{aligned} S : \dot{E} d\Omega_0 &= (JF^{-1}\boldsymbol{\sigma}F^{-T}) \, : \, (F^T D F) d\Omega_0 \\ &= tr\left(F^{-1}\boldsymbol{\sigma}F^{-T}F^T D F\right) J d\Omega_0 \\ &= tr\left(F^{-1}\boldsymbol{\sigma} D F\right) J d\Omega_0 \\ &= tr(\boldsymbol{\sigma} D)\, d\Omega_t \\ &= \boldsymbol{\sigma} : D d\Omega_t \end{aligned}$$

14.1. Surface integral for constant pressure .

If the pressure is constant (independent of location) the following lumped surface traction force needs to be computed:

$$F_{t,a} = \mathrm{p} \int_{\partial\Omega_t} \mathbf{n}(\mathbf{x}) N_a(\mathbf{x})\, \mathrm{dA}_t$$

It is:

$$\mathbf{n}(\mathbf{x})\mathrm{dA}_t = \sum_b \sum_c \left(\frac{\partial N_b}{\partial \alpha} \frac{\partial N_c}{\partial \beta} \right) \mathbf{x}_b \times \mathbf{x}_c \; d\alpha d\beta$$

And the result is put together as

$$F_{t,a} = p\int_{\partial\Omega_t} N_a(\mathbf{x})\frac{\partial \mathbf{x}}{\partial\alpha}\times\frac{\partial \mathbf{x}}{\partial\beta}\,d\alpha d\beta$$
$$= p\sum_b\sum_c\left(\int\int N_a\frac{\partial N_b}{\partial\alpha}\frac{\partial N_c}{\partial\beta}\,d\alpha d\beta\right)\mathbf{x}_b\times\mathbf{x}_c$$

14.2. Reduced kinematic description.

A simple example of a hyper elastic material is a Neo-Hookean material which has a linear stress-strain relationship but is valid also for large strains. The strain energy function is $\Psi(C) = \frac{1}{2}\mu(\mathrm{tr}C - 3)$. To make it an incompressible material, instead a modified Cauchy tensor that has always a determinante of 1 should be used, namely:

$$\overline{C} = J^{-\frac{2}{3}}C$$

and replace the strain energy function $\Psi(C)$ by $\overline{\Psi}(C) := \Psi(\overline{C})$. The second Piola tensor becomes:

$$S = \frac{\partial\overline{\Psi}(C)}{\partial C} + pJC^{-1}$$

To calculate this derivative is a bit tedious. First one obtaines, using $\mathbf{det}\,(C) = III_C = J^2$:

$$\frac{\partial III_C}{\partial C} = J^2C^{-1}$$

Thus,

$$\frac{\partial J^{-2/3}}{\partial C} = \frac{\partial III_C^{-1/3}}{\partial C} = -\frac{1}{3}J^{-2/3}C^{-1}$$

To calculate the derivative of the modified strain energy function, the chain rule is applied:

$$\frac{\partial\overline{C}_{ij}}{\partial C_{kl}} = \frac{\partial(J^{-2/3}C_{ij})}{\partial C_{kl}}$$
$$= \frac{\partial J^{-2/3}}{\partial C_{kl}}C_{ij} + J^{-2/3}\delta_{ik}\delta_{jl}$$
$$= \frac{\partial III_C^{-1/3}}{\partial C_{kl}}C_{ij} + III_C^{-1/3}\delta_{ik}\delta_{jl}$$
$$= -\frac{1}{3}III_C^{-4/3}\frac{\partial III_C}{\partial C_{kl}}C_{ij} + III_C^{-1/3}\delta_{ik}\delta_{jl}$$
$$= -\frac{1}{3}III_C^{-4/3}III_C(C^{-1})_{lk}C_{ij} + III_C^{-1/3}\delta_{ik}\delta_{jl}$$
$$= -\frac{1}{3}III_C^{-1/3}(C^{-1})_{lk}C_{ij} + III_C^{-1/3}\delta_{ik}\delta_{jl}$$
$$= III_C^{-1/3}(-\frac{1}{3}C_{ij}(C^{-1})_{lk} + \delta_{ik}\delta_{jl})$$

This can written more elegantly as:

$$\frac{\partial\overline{C}}{\partial C} = J^{-2/3}(\mathbb{i} - \frac{1}{3}C\otimes C^{-1}) \tag{C.3}$$

Where the components of $\mathfrak{i}$ are: $\mathfrak{i}_{ijkl} = \delta_{ik}\delta_{jl}$ It could also be written as follows:

$$\frac{\partial \overline{C}}{\partial C} = J^{-2/3}\mathfrak{i} - \frac{1}{3}\overline{C} \otimes C^{-1} \tag{C.4}$$

For notational simplicity the following reduced 2nd Piola tensor is defined:

$$\overline{S} := 2\frac{\partial \Psi(\overline{C})}{\partial \overline{C}} \tag{C.5}$$

Therefore the following expression for the 2nd Piola tensor is obtained:

$$S = 2\frac{\partial \Psi(\overline{C})}{\partial C} \tag{C.6}$$

$$= J^{-2/3}\left(\overline{S} - \frac{1}{3}\left(\overline{S} : C\right) C^{-1}\right) \tag{C.7}$$

$$= J^{-2/3}\,\overline{S} - \frac{1}{3}\left(\overline{S} : \overline{C}\right) C^{-1} \tag{C.8}$$

LEMMA 2. *The hydrostatic pressure field of modified elasticity potential disappears.*
Proof: The hydrostatic pressure p is:

$$\begin{aligned}
p &= \frac{1}{3}J^{-1}(S : C) \\
&= \frac{1}{3}J^{-1}J^{-2/3}(\overline{S} : C - \frac{1}{3}(\overline{S} : C)C^{-1} : C) \\
&= \frac{1}{3}J^{-5/3}(\overline{S} : C - \frac{1}{3}(\overline{S} : C) \cdot 3) \\
&= 0
\end{aligned}$$

□

15. Appendix D: Notes on linearization

In components:

$$\begin{aligned}
2\frac{\partial S_{kl}}{\partial C_{nm}} &= 2\frac{\partial (\overline{S}_{ij}\mathfrak{X}_{ijkl})}{\partial C_{nm}} \\
&= 2\frac{\partial \overline{S}_{ij}}{\partial \overline{C}_{uv}}\mathfrak{X}_{uvnm}\mathfrak{X}_{ijkl} + 2\overline{S}_{ij}\frac{\partial \mathfrak{X}_{ijkl}}{\partial C_{nm}} \\
&= \overline{\mathfrak{D}}_{ijuv}\mathfrak{X}_{uvnm}\mathfrak{X}_{ijkl} + 2\overline{S}_{ij}\frac{\partial \mathfrak{X}_{ijkl}}{\partial C_{nm}} \quad \text{with} \quad \overline{\mathfrak{D}} = 2\frac{\partial \overline{S}}{\partial \overline{C}}.
\end{aligned}$$

The first part of this is:

$$\begin{aligned}
&\overline{\mathfrak{D}}_{ijuv}\mathfrak{X}_{uvnm}\mathfrak{X}_{ijkl} \\
&= J^{-4/3}\,\overline{\mathfrak{D}}_{ijuv}\left(\delta_{un}\delta_{vm} - \frac{1}{3}C_{uv}C^{-1}_{nm}\right)\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C^{-1}_{kl}\right) \\
&= J^{-4/3}\,(\overline{\mathfrak{D}}_{ijnm} - \frac{1}{3}\overline{\mathfrak{D}}_{ijuv}C_{uv}C^{-1}_{nm})\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C^{-1}_{kl}\right) \\
&= J^{-4/3}\left(\overline{\mathfrak{D}}_{klnm} - \frac{1}{3}(\overline{\mathfrak{D}}_{kluv}C_{uv}C^{-1}_{nm} + \overline{\mathfrak{D}}_{ijnm}C_{ij}C^{-1}_{kl}) + \frac{1}{9}\overline{\mathfrak{D}}_{ijuv}C_{uv}C_{ij}C^{-1}_{kl}C^{-1}_{nm}\right) \quad .
\end{aligned}$$

In tensor notation this is:

$$\overline{\mathfrak{D}} : \mathfrak{X} : \mathfrak{X} =$$

$$J^{-4/3}\left[\overline{\mathfrak{D}} - \frac{1}{3}\left((\overline{\mathfrak{D}} : C) \otimes C^{-1} + C^{-1} \otimes (C : \overline{\mathfrak{D}})\right) + \frac{1}{9}(C : \overline{\mathfrak{D}} : C) C^{-1} \otimes C^{-1}\right]$$

The other part becomes on expansion:

$$\begin{aligned}
\frac{\partial \mathfrak{X}_{ijkl}}{\partial C_{nm}} &= \frac{\partial}{\partial C_{nm}}\left(J^{-2/3}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C_{kl}^{-1}\right)\right) = \\
&= -\frac{1}{3}J^{-2/3}C_{nm}^{-1}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C_{kl}^{-1}\right) + J^{-2/3}\frac{\partial}{\partial C_{nm}}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C_{kl}^{-1}\right) \\
&= -\frac{1}{3}J^{-2/3}\left[C_{nm}^{-1}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C_{kl}^{-1}\right) + \frac{\partial(C_{ij}C_{kl}^{-1})}{\partial C_{nm}}\right] \\
&= -\frac{1}{3}J^{-2/3}\left[C_{nm}^{-1}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}C_{ij}C_{kl}^{-1}\right) + \delta_{in}\delta_{jm}C_{kl}^{-1} - C_{ij}C_{kn}^{-1}C_{lm}^{-1}\right] \\
&= -\frac{1}{3}J^{-2/3}\left[\delta_{ik}\delta_{jl}C_{nm}^{-1} + \delta_{in}\delta_{jm}C_{kl}^{-1} - C_{ij}C_{kn}^{-1}C_{lm}^{-1} - \frac{1}{3}C_{ij}C_{kl}^{-1}C_{nm}^{-1}\right]
\end{aligned}$$

The contraction of this long expression with $\overline{S}_{ij}$ yields after some work:

$$-\frac{1}{3}J^{-2/3}\left[\overline{S}_{kl}C_{nm}^{-1} + \overline{S}_{nm}C_{kl}^{-1} - (\overline{S}_{ij}C_{ij})\left(C_{kn}^{-1}C_{lm}^{-1} - \frac{1}{3}C_{kl}^{-1}C_{nm}^{-1}\right)\right]$$

This can be rewritten in tensor notation:

$$\overline{S} : \frac{\partial \mathfrak{X}}{\partial C} = -\frac{1}{3}J^{-2/3}\left[\overline{S} \otimes C^{-1} + C^{-1} \otimes \overline{S} - (\overline{S} : C)(\mathfrak{I} - \frac{1}{3}C^{-1} \otimes C^{-1})\right] ,$$

where

$$\mathfrak{I}_{ijkl} = C_{ik}^{-1}C_{jl}^{-1} .$$

Finally, the modified elasticity tensor is the following:

$$\begin{aligned}
2\frac{\partial S}{\partial C} &= \\
&J^{-4/3}\left[\overline{\mathfrak{D}} - \frac{1}{3}\left((\overline{\mathfrak{D}} : C) \otimes C^{-1} + C^{-1} \otimes (C : \overline{\mathfrak{D}})\right) + \frac{1}{9}(C : \overline{\mathfrak{D}} : C) C^{-1} \otimes C^{-1}\right] \\
&- \frac{2}{3}J^{-2/3}\left[\overline{S} \otimes C^{-1} + C^{-1} \otimes \overline{S} - (\overline{S} : C)(\mathfrak{I} - \frac{1}{3}C^{-1} \otimes C^{-1})\right]
\end{aligned}$$

16. Appendix E: Notes related to fluid mechanics

LEMMA 3. *If the kinematic viscosity ν is constant, the double divergence of the viscous term in the Navier Stokes equations of an incompressible flow disappears:*
Proof via index representation, using $v_{i,i} = 0$.

$$\begin{aligned}
div\, div D = D_{ij,ji} &= \frac{1}{2}(v_{i,j} + v_{j,i})_{,ji} \\
&= \frac{1}{2}(v_{i,jj} + v_{j,ij})_{,i} = \frac{1}{2}(v_{i,jj} + 0)_{,i} \\
&= v_{i,jji} = v_{i,ijj} = 0
\end{aligned}$$

LEMMA 4. *If* $div\mathbf{v} = 0$, *it follows that:* $div(\nabla\mathbf{v}(\mathbf{v}-\mathbf{w})) = tr(\nabla\mathbf{v}\nabla(\mathbf{v}-\mathbf{w}))$
Proof:

$$(v_{i,j}(v_j - w_j))_{,i} = v_{i,ji}(v_j - w_j) + v_{i,j}(v_{j,i} - w_{j,i}) = v_{i,j}(v_{j,i} - w_{j,i})$$

LEMMA 5. *If* $\nabla\mathbf{v} = (\nabla\mathbf{v})^T$, *the velocity field is free of rotation and can be represented as the gradient of a velocity potential. In this case, an inhomogeneous Bernoulli equation for the velocity potential is obtained.*
First it is clear that:

$$\nabla\mathbf{v}\cdot\mathbf{v} = \nabla\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right) \quad \textit{if} \quad \nabla\mathbf{v} = (\nabla\mathbf{v})^T <=> \mathrm{rot}\mathbf{v} = 0$$

Also, if $\nabla\mathbf{v}$ *is symmetric, it can be written:*

$$\frac{1}{2}\nabla[(\mathbf{v}+\mathbf{w})(\mathbf{v}-\mathbf{w})] = \frac{1}{2}\nabla[\mathbf{v}^2 - \mathbf{w}^2] = \nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{w}) + (\nabla\mathbf{w})^T(\mathbf{v}-\mathbf{w})$$

Using this to replace the term $\nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{w})$ *in the moment equation, the following Bernoulli type equation is obtained:*

$$\mathbf{v}' + \frac{1}{2}\nabla[\mathbf{v}^2 - \mathbf{w}^2] + \frac{1}{\rho_0}\nabla p = (\nabla\mathbf{w})^T(\mathbf{v}-\mathbf{w}) \tag{E.1}$$

If the velocity potential is ϕ, *this can be rewritten as:*

$$\nabla\left(\phi' + \frac{1}{2}[(\nabla\phi)^2 - \mathbf{w}^2] + \frac{p}{\rho_0}\right) = (\nabla\mathbf{w})^T(\nabla\phi - \mathbf{w}) \tag{E.2}$$

17. Appendix F: Implicite time stepping method

A general implicite time stepping method is described here that can be used for solving the discrete equations of the vocal tract model. For the sake of generality, the variables $\underline{x}$, $\underline{p}$ and others relating to the discrete model state are replaced by a vector variable q. The augmented virtual work equation can then be written in the following general form:

$$0 = r(q) = M\ddot{q} + f(q,\dot{q}) - g(q,t) \tag{F.1}$$

This needs to be linearized around an iteration point q_{n+1}^k, whereby the upper index represents local iteration (time frozen), and the lower index represents time.

$$r(q_{n+1}^{k+1}) = r(q_{n+1}^k) + S(q_{n+1}^k)\,(q_{n+1}^{k+1} - q_{n+1}^k) \tag{F.2}$$

where

$$S(q_{n+1}^k) = \frac{\partial r}{\partial q}\Big|_{q_{n+1}^k} \tag{F.3}$$

We got:

$$S(q) = M\frac{\partial\ddot{q}}{\partial q} + \frac{\partial f}{\partial q} + \frac{\partial f}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial q} - \frac{\partial g}{\partial q} \tag{F.4}$$

In the well-known Newmark method, the following approximations are used:

$$\frac{\partial\ddot{q}}{\partial q} = \frac{1}{\beta h^2}id \quad \text{and} \quad \frac{\partial\dot{q}}{\partial q} = \frac{\gamma}{\beta h}id$$

whereby usually $\beta = 1/4$ and $\gamma = 1/2$, id is the unit matrix of dimension equal to that of q.

The dynamic equilibrium is computed by solving the equation $r(q_{n+1}^{k+1}) = 0$ by means of a Newton iteration. During the Newton iteration the time is "frozen", that is, the lower index of q does not change. At the end of the iteration (if it converges) a new location q_{n+1} has been found. Then one step in the Newmark algorithm is computed, which results in a new (predicted) velocity, and acceleration. After that a new Newton iteration is executed that will result in a new state, etc.

17.1. Newmark's method.

The well-known Newmark's method is based on the following approximations of the integrals over a time-interval h.

$$\int_{t_n}^{t_{n+1}} \ddot{q}(\tau)d\tau \approx \quad (1-\gamma)h\ddot{q}_n + \gamma h\ddot{q}_{n+1}$$

$$\int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)\ddot{q}(\tau)d\tau \approx (\frac{1}{2}-\beta)h^2\ddot{q}_n + \beta h^2\ddot{q}_{n+1}$$

If the coefficients in these approximations are chosen as $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, the resulting time stepping algorithm is unconditionally stable for the linear case. The algorithm is based on the idea:

Given q_n, $\dot{q}_n$, and $\ddot{q}_n$, express $\dot{q}_{n+1}$, and $\ddot{q}_{n+1}$ by q_n, $\dot{q}_n$, $\ddot{q}_n$, and the unknown q_{n+1}. Insert this into the linearized dynamic equilibrium equation, nd find q_{n+1} by iteration.

Using Newmark's integral approximations, the following is obtained:

$$q_{n+1} = q_n + h\dot{q}_n + (\frac{1}{2}-\beta)h^2\ddot{q}_n + \beta h^2\ddot{q}_{n+1}$$
$$\dot{q}_{n+1} = \dot{q}_n + (1-\gamma)h\dot{q}_n + \gamma h\ddot{q}_{n+1}$$

From this one through elimination it can be found:

$$\text{(F.5)} \qquad \begin{bmatrix} \dot{q}_{n+1} \\ \ddot{q}_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{\gamma}{\beta} & (1-\frac{\gamma}{2\beta})h \\ -\frac{1}{\beta h} & 1-\frac{1}{2\beta} \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_n \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} \frac{\gamma}{\beta h} \\ \frac{1}{\beta h^2} \end{bmatrix} \cdot (q_{n+1}-q_n) \quad .$$

This equation must not be understood as a 2-dimensional matrix equation but as a $2n$-dimensional equation. All variables are n-dimensional vectors. For the special choice of $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, it becomes:

$$\text{(F.6)} \qquad \begin{bmatrix} \dot{q}_{n+1} \\ \ddot{q}_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -\frac{4}{h} & -1 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_n \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} \frac{2}{h} \\ \frac{4}{h^2} \end{bmatrix} \cdot (q_{n+1}-q_n) \quad .$$

For every approximation of q_{n+1}, which is obtained at each step during the iteration of the dynamic equilibrium equation as q^k_{n+1}, the variables $\dot{q}^k_{n+1}$ and $\ddot{q}^k_{n+1}$ are computed via the above matrix equation (F.5).

18. Appendix G: Some relevant constants and definitions

Viscosity of air: $\mu = 1.84 \cdot 10^{-4} \frac{\mathrm{g}}{\mathrm{cm\ s}}$
Density of air: $\rho_0 = 1.2 \cdot 10^{-3} \frac{\mathrm{g}}{\mathrm{cm}^3}$
Kinematic viscosity of air: $\nu = \frac{\mu}{\rho_0} = 0.153 \frac{\mathrm{cm}^2}{\mathrm{s}}$
Fluid mechanically relevant dimension free numbers:
Reynolds number: $Re = \frac{v_0 L_0 \rho_0}{\mu}$
Strouhal number: $St = \frac{L_0}{v_0 t_0}$

18.1. List of frequently used symbols.

$\boldsymbol{\sigma}$	(spatial) Cauchy stress tensor
ρ, ρ_0	Density
$S, \overline{S}$	(material) 2nd Piola stress tensor
$\mathbf{p}$	material coordinate (reference)
$\mathbf{x}$	particle coordinate
$\mathbf{v}, \mathbf{a}$	velocity, accelaration field.
$\mathbf{w}$	grid velocity field
$\mathbf{v}', \dot{\mathbf{v}}$	local time derivative, spatial and material
f	force field, various use
$\delta\mathbf{v}$	variational velocity field
p, π	pressure variables or hydrostatic pressure
N_a, Q_a, P_a, R_a	shape or interpolation functions
δW	virtual work
F	deformation gradient
J	Jacobian $\det F$
I	2nd order unity tensor δ_{ij}
C	Cauchy deformation tensor
E	Lagrange deformation tensor
D	symmetric part of velocity gradient
$\mathbf{n}$	surface normal
$\mathbf{x} \cdot \mathbf{y}$	inner product of two vectors $(x_i y_i)$ summation over i
$\mathbf{x} \otimes \mathbf{y}$	outer product of two vectors $(x_i y_j)$
$A : B$	2nd order tensor contraction $(A_{ij} B_{ij})$.

Bibliography

[1] P. K. Banerjee. *The boundary element methods in engineering.* McGraw-Hill, 1994.

[2] K.-J. Bathe. *Finite Element Procedures.* Prentice Hall, 1996.

[3] J. Bonet and R. D. Wood. *Nonlinear continuum mechanics for finite element analysis.* Cambridge University Press, 1997.

[4] C. Canuto. Spectral methods. In *Computational fluid dynamics*, chapter 9, pages 443–500. Elsevier, 1996.

[5] J. Dang and K. Honda. A physiological model of a dynamic vocal tract for speech production. Technical Report 43.70.Aj, 43.70.Bk, ATR Human Information Processing Research Labs, 1998.

[6] A. P. Dowling and J. E. Ffowcs Williams. *Sound and sources of sound.* Ellis Horwood Limited - John Wiley & Sons, 1983.

[7] G. R. Farley. A biomechanical laryngeal model of voice f_0 and glottal width control. *J. Acoust. Soc. Am.*, 100(6), 1996.

[8] J. H. Ferziger and M. Perić. *Computational Methods for Fluid Mechanics.* Springer, 2nd edition, 1997.

[9] B. R. Fink and R. J. Demarest. *Laryngeal Biomechanics.* Harvard University Press, 1978.

[10] Morton E. Gurtin. *An Introduction to Continuum Mechanics.* Academic Press, New York, 1981.

[11] R. S. McGowan. The quasisteady approximation in speech production. *J. Acoust. Soc. Am.*, 94(5):3011–3013, 1993.

[12] X. Pelorson, A. Hirschberg, R. R. van Hassel, A. P. J. Wijnands, and Y. Auregan. Theoretical and experiemtal study of quasisteady-flow separation within the glottis during phonation. application to a modfied two-mass model. *J. Acoust. Soc. Am.*, 96(6):3416–3431, 1994.

[13] B. H. Story and I. R. Titze. Voice simulation with a body-cover model of the vocal folds. *J. Acoust. Soc. Am.*, 97(2):1249–1260, 1995.