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Abstract

This paper provides a study on the structure-from-motion problem under the conditions that two planes independently and rigidly move in three dimensions and that, for given two perspective images, the correspondences of points in the planes are known. A typical approach to this problem involves segmenting the images into regions each of which has only one plane, and then determining the normal vector of each plane and the motion of the plane. In this paper, however, we take a different approach where the images need not be segmented: we directly handle the images in which two planes exist. We show that we can generally determine the normal vectors of the two planes and the two motions when we observe 17 points, where the tensor product of two transformation matrices and its decomposition play the central role. We first determine the tensor product and then decompose it into the two transformation matrices. Here the tensor product is expressed as a pair of its symmetric part and its alternating part. We also clarify the cases where the tensor product cannot be determined. It is shown that when the two planes share the same rotation, we cannot determine the alternating part if the two normal vectors are parallel or the two translation vectors are parallel. Furthermore, we show that unless at least four points exist in each plane such that no three of them are collinear, we cannot determine the symmetric part; whereas at least seven points are needed in each plane to determine the alternating part.

Key Words: structure from motion, planes, transformation matrix, tensor product, symmetrization, alternization, critical conditions.

Contents

1 Introduction

The importance of motion estimation in computer vision has long been emphasized. There is a long list of applications in sensing, modeling, and interpretation of motion and structure from feature correspondences among images observed at different times. It includes target tracking, vehicle navigation, robot guidance and the monitoring of dynamic industrial processes. As a result, the structure-from-motion problem has been studied in several paradigms [1], [5], [19], [20], and a number of algorithms for this problem have been proposed. Some are feature-based and some are optical-flow-based. In this paper, we focus on the feature-based approach.

In the feature-based approach, two or more images are used for computing structure and motion of a point set in 3-D; the correspondences of points among the images are assumed to be known. Two cases are distinguished to apply a structure-from-motion algorithm: i) points in 3-D are coplanar and ii) they are noncoplanar. The first case is equivalent to handling a plane in 3-D; whereas the second one is equivalent to handling general points in 3-D.

As for structure and motion of a plane, Tsai–Huang [15] defined the eight "pure parameters" in terms of the normal vector of a plane and motion parameters, and showed that the pure parameters can be linearly computed from eight point correspondences in two images and from them the motion parameters can be computed by solving a sixth-order polynomial of one unknown only: the number of solutions never exceeds two. Tsai–Huang [18] improved the computational task. This showed that motion parameters can be linearly computed from four point correspondences in two images, and proposed a linear algorithm where singular value decomposition of a matrix consisting of the eight pure parameters, called the *transformation matrix* in this paper, plays the central role. Longuet-Higgins [10] showed the same results in a different form. It also analyzed the ambiguity incurred by a special relation between the two viewpoints. Tsai–Huang [17] further extended the results to the case where three images are available, showing the uniqueness of the solution.

As for general points in 3-D, Longuet-Higgins [8] showed that we can determine structure and motion from eight point correspondences in two images, and proposed a linear algorithm, called the 8-point algorithm, for computing structure and motion. Longuet-Higgins [9] showed that points in a special configuration, called a *critical surface*, do not allow us to determine the structure and motion. Tsai–Huang [16] independently proposed a similar algorithm based on singular value decomposition. Zhuang–Huang–Haralick [22] unified the two algorithms in terms of the essential matrix. The case where more than eight points in two images are available, was investigated in [21]; and an algorithm based on the method of least squares was proposed to obtain the optimal solution. Lee [7] treated points in a special configuration to reduce the number of required points, showing that we can determine structure and motion from six point correspondences in two images such that four of the six points are coplanar. For cases where more images are available, a factorization method [11], [14] was proposed under orthographical/paraperspective projection. Here also, singular value decomposition of a matrix, which is derived from a number of images, plays the central role. Christy–Horaud [3] proposed a method for solving the Euclidean reconstruction problem with a perspective camera by incrementally performing a Euclidean reconstruction with the factorization method.

These algorithms are all based on the assumption that only a single object exists in the images. In other words, the segmentations are already executed in stages prior to using the algorithm. However, as pointed out in [2], [6] or [13], segmentation is one of the most crucial problems in computer vision. To avoid the segmentation problem, we must directly handle images in which plural objects exist, without knowing which feature point belongs to which object in the image. Costeira–Kanade [4] proposed a method, in the factorization scheme, for segmenting and recovering the motion and shape of multiple independently moving objects from a set of feature trajectories tracked in a sequence of images. However, it is assumed in the method that the projection is orthographically performed. Shizawa [12] investigated the case where two (general) 3-D objects exist in the images, showing that we can generally determine the structures of the two objects and the two motions when we observe 35 point correspondences in two perspective images. In line with Shizawa [12], in this paper we investigate the case where two planes exist in the perspective images.

This paper addresses the study on the structure-from-motion problem under the conditions that two planes independently and rigidly move in 3-D and that, for given two perspective images, the correspondences of points in the planes are known. We show that we can generally determine the normal vectors of the two planes and the two motions when we observe 17 points, where the tensor product (in the sense of tensor algebra) of two transformation matrices and its decomposition play the central role. We first determine the tensor product and then decompose it into the two transformation matrices. Singular value decomposition of a transformation matrix leads to determining (almost uniquely) the normal vector of a plane and the motion parameters. The tensor product is expressed as a pair of its symmetric part and its alternating part. The symmetric part has 36 independent unknowns and each point correspondence gives three linear homogeneous constraint equations to them. In contrast, the alternating part has 18 independent unknowns and each point correspondence gives one linear homogeneous constraint equation to them. Therefore, in general, we can uniquely determine each part up to a scale factor with linear computation. We then give a procedure to decompose the symmetric part and the alternating part together into two transformation matrices. This procedure makes full use of constraint equations to determine the two transformation matrices. Using all of the constraint equations leads to a computationally simple procedure as well as a robust one with respect to noise. The decomposition procedure is divided into two steps: one is determining each column vector of the transformation matrices up to an unknown scale factor and the other is determining the scale factors there to determine the transformation matrices. Here the ratio of the scale factors incurred in determining the symmetric part and the alternating part, plays an important role in uniquely determining the transformation matrices. We also clarify in what cases we cannot determine the symmetric part or the alternating part. It is shown that when two planes share the same rotation, we cannot determine the alternating part if the normal vectors of the two planes are parallel or if the translation vectors of the two planes are parallel. We then show that unless we have at least four points in each plane such that no three of them are collinear, we cannot determine the symmetric part; whereas at least seven points are needed in each plane to determine the alternating part. Accordingly, now we can directly handle images in which two planes exist; we can determine motions and structures of the two planes without executing segmentation.

This paper is organized as follows. In Section 2 we briefly review a case concerning only one plane. Here we introduce a *transformation matrix* that is derived from the normal vector of a plane and its motion parameters. In Section 3 we derive constraint equations on the coordinates of two images of a point that exists in one of two planes, where the tensor product of transformation matrices plays the central role. The tensor product is decomposed into its symmetric part and its alternating part; each part is investigated separately where the symmetrization and the alternization enable us to reduce the number of unknowns. We then pay attention to the conditions, which we call the *critical conditions*, where we cannot determine the symmetric part or the alternating part, and clarify them. Two cases are distinguished: the case where two planes themselves do not allow us to determine the symmetric part or the alternating part and, the other case where points in a special configuration do not allow us to do so. In Section 4 we give a procedure to decompose the symmetric part and the alternating part together into two transformation matrices. In Section 5 we present our algorithm for computing transformation matrices from given point correspondences. In this paper, we assume that two planes independently and rigidly move around a fixed view point and that the correspondences of points in the planes between two images are known.

2 Projection and plane transformation

In preparation for further investigation, here we briefly review a case where only one plane is concerned.

2.1 Projection into image plane

Let us consider a calibrated pinhole camera model with focal length f. Then we may assume that the camera performs a perspective projection whose origin O coincides with the center of the lens and whose Z-axis is aligned with the optical axis; $Z = f$ is the image plane. In this model[1], the coordinates $\boldsymbol{X} = (X, Y, Z)^{\mathrm{T}}$ of a point in 3-D are projected to $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y})^{\mathrm{T}}$ in the image plane, where

$$\tilde{x} = f\frac{X}{Z}, \qquad \tilde{y} = f\frac{Y}{Z}.$$

We embed the image plane in $\mathcal{P}^2$, the projective plane over the real number field $\mathbf{R}$, so that the Euclidean coordinates $\tilde{\boldsymbol{x}}$ are expressed by the homogeneous coordinates $\boldsymbol{x} = (\tilde{x}, \tilde{y}, 1)^{\mathrm{T}}$. We then obtain the relation between Euclidean coordinates $\boldsymbol{X}$ of a point in 3-D and the homogeneous coordinates $\boldsymbol{x}$ of its image in $\mathcal{P}^2$:

$$\boldsymbol{x} \;=\; \iota \boldsymbol{X} \qquad (\iota \in \mathbf{R}^*),$$

[1]We use a column vector and denote by $\boldsymbol{x}^{\mathrm{T}}$ the transportation of a vector $\boldsymbol{x}$.

where ι depends on the point and its value is unknown. $1/\iota$ is sometimes referred to as the *depth* of the point. Note that $\mathbf{R}^*$ denotes the set of non-zero real numbers. Accordingly, by embedding the image plane in $\mathcal{P}^2$, we can directly handle the Euclidean coordinates of a point with an unknown scale factor. Henceforth, if not explicitly stated, the coordinates of a point in the image plane are understood to be homogeneous.

2.2 Plane transformation

Let a plane in 3-D be

$$\boldsymbol{n} \cdot \boldsymbol{X} = 1. \tag{2.1}$$

In this paper, we call $\boldsymbol{n}$ the *normal vector* of the plane. We suppose that a point in plane (2.1) is subject to a rigid motion as follows:

$$\boldsymbol{Y} = R\boldsymbol{X} + \boldsymbol{t}, \tag{2.2}$$

where R is a rotation matrix and $\boldsymbol{t}$ is a translation vector. Here $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the Euclidean coordinates of a point before and after a motion, respectively. Combining (2.1) with (2.2), we obtain

$$\boldsymbol{Y} = \left(R + \boldsymbol{t}\boldsymbol{n}^{\mathrm{T}}\right) \boldsymbol{X}.$$

Hence, for the homogeneous coordinates of two images (before and after the motion) of a point in plane (2.1), we have

$$\boldsymbol{y} = \kappa \left(R + \boldsymbol{t}\boldsymbol{n}^{\mathrm{T}}\right) \boldsymbol{x} \qquad (\kappa \in \mathbf{R}^*). \tag{2.3}$$

We remark again that κ here depends on the point; its value is unknown. Defining a 3×3 matrix M, referred to as the *transformation matrix* in this paper,

$$M = R + \boldsymbol{t}\boldsymbol{n}^{\mathrm{T}},$$

and hiding the scale factor κ, we can rewrite (2.3) as

$$[\boldsymbol{y}]\, M \boldsymbol{x} = 0, \tag{2.4}$$

since neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is a zero vector. Here, for $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ in general, $[\,\boldsymbol{x}\,]$ is defined by

$$[\,\boldsymbol{x}\,] \ := \ \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

The homogeneous coordinates of two images (before and after a motion) of a point in a plane satisfy (2.4); this is the constraint equation when only one plane is concerned. The transformation matrix M has nine parameters; (2.4) are linear homogeneous equations with them. As we can see, each point in a plane gives two independent constraint equations in (2.4). Accordingly, with four point correspondences, in general, we can uniquely determine M up to a scale factor. Once M is known, we can (almost uniquely) determine [18] the rotation matrix R, the translation vector $\boldsymbol{t}$ and the normal vector $\boldsymbol{n}$ of the plane, where singular value decomposition of M plays the central role (depending on the number of different singular values of M, we have indeterminacy in recovering them). We focus on (2.4) to investigate the case where two planes exist in 3-D.

Remark 2.1 The length of $\boldsymbol{n}$ does make sense since we set the constant term of a plane equation to be 1. We can also discuss the case where a plane is expressed in another form; the results below hold true. We use plane equation (2.1) in line with [15], [18] and [17]. □

3 Constraint equations derived from two moving planes

In this section, we derive constraint equations on the coordinates of two images (before and after a motion) of a point in one of two planes, where the tensor product plays the central role. The equations are linear with respect to independent parameters; the parameters together form a covariant tensor of degree 2. We thus decompose the tensor into its symmetric part and its alternating part; we then discuss each part separately.

3.1 Tensor product constraint

We assume that two planes 1 and 2 independently and rigidly move in three dimensions where a motion of a plane is characterized as that of a point in the plane (see Fig. 1). Let the transformation matrices of planes 1 and 2 be M^1 and M^2, respectively. Then, the coordinates

$\boldsymbol{x}$ and $\boldsymbol{y}$ of two images (before and after a motion) of a point in plane 1 satisfy

$$[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x} \;=\; 0. \tag{3.1}$$

For a point in plane 2, the coordinates of its two images before and after a motion satisfy

$$[\,\boldsymbol{y}\,]\,M^2\boldsymbol{x} \;=\; 0. \tag{3.2}$$

Accordingly, when we observe two images (before and after a motion) of a point that exists on plane 1 or 2, their coordinates satisfy (3.1) or (3.2), which is expressed by

$$[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x}\otimes[\,\boldsymbol{y}\,]\,M^2\boldsymbol{x} \;=\; O. \tag{3.3}$$

Here, for any linear spaces L_1 and L_2, $L_1\otimes L_2$ denotes the tensor product of L_1 and L_2. Our aim in this paper is, for given correspondence pairs of $\{\boldsymbol{x},\boldsymbol{y}\}$, to determine M^1 and M^2 in (3.3).

It is essential that we regard both $\boldsymbol{x}$ and $\boldsymbol{y}$ in (3.3) as contravariant vectors, and both M^1 and M^2 as covariant vectors. Therefore, (3.3) is expressed in the form of

$$\sum_{\lambda=1}^{9}\sum_{\mu=1}^{9}\mathit{\Gamma}_{\lambda\mu}\,\zeta^{\lambda}\,\eta^{pq\mu} \;=\; 0 \qquad (p,q\in\{1,2,3\}),$$

where $\mathit{\Gamma}_{\lambda\mu}$ is a covariant tensor of degree 2, and both ζ^{λ} and $\eta^{pq\mu}$ are contravariant tensors of degree 2. It should be noted that $\mathit{\Gamma}_{\lambda\mu}$ (as the tensor product of M^1 and M^2) are unknown parameters whose number is 81, whereas ζ^{λ} (as the tensor product of $\boldsymbol{x}$'s) and $\eta^{pq\mu}$ (as the tensor product of $\boldsymbol{y}$'s) are both known; the equation is linear and homogeneous in $\mathit{\Gamma}_{\lambda\mu}$. Since any tensor of degree 2 can be uniquely expressed as the sum of its symmetric part and its alternating part, the tensor above is decomposed into its symmetric part and its alternating part. Namely, instead of directly handling (3.3), we use the following:

$$\begin{cases}\text{the symmetric part of the LHS of (3.3)} &= \;O\\ \text{the alternating part of the LHS of (3.3)} &= \;O.\end{cases}$$

Here LHS stands for the left-hand-side. We denote by $\mathcal{S}$ and $\mathcal{A}$ the symmetric part and the alternating part of the LHS of (3.3), respectively. It is important to note that for each point, (3.3) gives nine equations, six of which are possessed by $\mathcal{S}$; the other three are possessed by $\mathcal{A}$. We also remark that 45 of the 81 unknown parameters $\mathit{\Gamma}_{\lambda\mu}$ appear in $\mathcal{S}$ and the other 36 appear in $\mathcal{A}$.

3.2 Symmetric part of the tensor product

The symmetric part $\mathcal{S}$ of the LHS of (3.3) is expressed by

$$\mathcal{S} = \frac{1}{2}\Big\{[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x}\otimes[\,\boldsymbol{y}\,]\,M^2\boldsymbol{x}+[\,\boldsymbol{y}\,]\,M^2\boldsymbol{x}\otimes[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x}\Big\}. \tag{3.4}$$

Defining

$$T_{ijk\ell} := M^1_{ik}\,M^2_{j\ell} \qquad (i,j,k,\ell\in\{1,2,3\}),$$

we then obtain the (p,q)-component[2], denoted by $\mathcal{S}^{pq}$, of $\mathcal{S}$:

$$\mathcal{S}^{pq} = T_{ijk\ell}\left(\frac{[\,\boldsymbol{y}\,]^{pi}\,[\,\boldsymbol{y}\,]^{qj}+[\,\boldsymbol{y}\,]^{pj}\,[\,\boldsymbol{y}\,]^{qi}}{2}\right)x^k\,x^\ell \qquad (p,q\in\{1,2,3\}). \tag{3.5}$$

where M^{ij} or M_{ij} denotes the (i,j)-component of a matrix M and x^k denotes the k-th component of a vector $\boldsymbol{x}$. Note that $\mathcal{S}^{pq}=\mathcal{S}^{qp}$. A superscript index is used for a contravariant vector; a subscript index is used for a covariant vector. In (3.5), we use Einstein summation convention: a repeated index, which appears both "above" and "below", implies the summation from 1 to 3. This is applied to equations below. We can see that the indices i and j in (3.5) are symmetric to each other since exchanging i and j results in the same term in $\boldsymbol{x}$ and $\boldsymbol{y}$. We can also see that the indices k and ℓ in (3.5) are symmetric to each other. Therefore, we may symmetrize i and j, and k and ℓ, respectively, which yields

$$\mathcal{S}^{pq} = \psi(i,j,k,\ell)\,T_{(i\,j)(k\,\ell)}\,[\,\boldsymbol{y}\,]^{p(i}\,[\,\boldsymbol{y}\,]^{|q|j)}\,x^{(k}x^{\ell)}.$$

Here $(\cdots)$ implies the symmetrization of the indices there except for those in $|\cdots|$; $T_{(i\,j)(k\,\ell)}$ is, thus, defined by

$$T_{(i\,j)(k\,\ell)} := \frac{1}{2!\cdot 2!}\left(T_{ijk\ell}+T_{jik\ell}+T_{ij\ell k}+T_{ji\ell k}\right).$$

$\psi(i,j,k,\ell)$ is a function such that

$$\psi(i,j,k,\ell) := \begin{cases} 1 & \text{if } i=j \text{ and } k=\ell \\ 4 & \text{if } i\neq j \text{ and } k\neq \ell \\ 2 & \text{otherwise.} \end{cases}$$

[2]The (p,q)-component of $L_1\otimes L_2$ is the product of the p-th component of L_1 and the q-th of L_2.

Accordingly, the condition that the symmetric part of the LHS of (3.3) $= O$, reduces to

$$\psi(i,j,k,\ell)\, T_{(i\,j)(k\,\ell)}\, [\,\boldsymbol{y}\,]^{p(i}\, [\,\boldsymbol{y}\,]^{|q|j)}\, x^{(k} x^{\ell)} \;=\; 0 \qquad (p \le q\,;\; p,q \in \{1,2,3\}). \tag{3.6}$$

Here $T_{(i\,j)(k\,\ell)}$ are independent unknowns whose number is 36; (3.6) are linear homogeneous equations with $T_{(i\,j)(k\,\ell)}$. For each point correspondence, we have six cases in choosing p and q in (3.6); however, the following theorem shows that only three of them are independent. We thus can, in general, uniquely determine $T_{(i\,j)(k\,\ell)}$ up to a scale factor with linear computation if we have 12 point correspondences. It is important to remark that the symmetrization of i and j, and that of k and ℓ reduced the number of independent unknowns from 45 to 36.

Theorem 3.1 Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{R}^3$ and

$$\Theta \;=\; \frac{1}{2}\{[\,\boldsymbol{u}\,]\,\boldsymbol{v} \otimes [\,\boldsymbol{u}\,]\,\boldsymbol{w} + [\,\boldsymbol{u}\,]\,\boldsymbol{w} \otimes [\,\boldsymbol{u}\,]\,\boldsymbol{v}\}\,.$$

We denote by Θ_{pq} the (p,q)-component of Θ $(p,q \in \{1,2,3\})$. Then, we have $\Theta_{pq} = \Theta_{qp}$. Furthermore, $\Theta_{pq} = 0$ gives only three independent equations.

Proof: $\Theta_{pq} = \Theta_{qp}$ is obvious from the definition. Letting $\boldsymbol{f} = [\,\boldsymbol{u}\,]\,\boldsymbol{v}$ and $\boldsymbol{g} = [\,\boldsymbol{u}\,]\,\boldsymbol{w}$, we have

$$\boldsymbol{u} \cdot \boldsymbol{f} = 0, \qquad \boldsymbol{u} \cdot \boldsymbol{g} = 0.$$

Hence the third components of $\boldsymbol{f}$ and $\boldsymbol{g}$ are expressed as a linear combination of its first component and second one, respectively, with common coefficients:

$$f_3 = \alpha\, f_1 + \beta\, f_2, \qquad g_3 = \alpha\, g_1 + \beta\, g_2.$$

We then have

$$\begin{aligned} \Theta_{33} &= f_3\, g_3 \\ &= (\alpha\, f_1 + \beta\, f_2)\,(\alpha\, g_1 + \beta\, g_2) \\ &= \alpha^2\, \Theta_{11} + 2\alpha\beta\, \Theta_{12} + \beta^2\, \Theta_{22}. \end{aligned}$$

In the same way, Θ_{23} and Θ_{13} are respectively expressed as a linear combination of Θ_{11}, Θ_{22} and Θ_{12}. Accordingly, nine equations $\Theta_{pq} = 0$ $(p,q \in \{1,2,3\})$ reduce to only three independent ones. □

3.3 Alternating part of the tensor product

We now turn to the alternating part $\mathcal{A}$ of the LHS of (3.3). $\mathcal{A}$ is expressed by

$$\mathcal{A} \;=\; \frac{1}{2}\Big\{[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x}\otimes[\,\boldsymbol{y}\,]\,M^2\boldsymbol{x} - [\,\boldsymbol{y}\,]\,M^2\boldsymbol{x}\otimes[\,\boldsymbol{y}\,]\,M^1\boldsymbol{x}\Big\}. \tag{3.7}$$

We investigate $\mathcal{A}$ from a point of view different from that in the previous section. Namely, we discuss $\mathcal{A}$ in the context of exterior algebra. For $\boldsymbol{u}$, $\boldsymbol{v} \in \mathrm{R}^3$, their exterior product $\boldsymbol{u} \wedge \boldsymbol{v}$ is related to their tensor product as follows:

$$\boldsymbol{u} \wedge \boldsymbol{v} \;=\; \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}.$$

By using the exterior product, we can rewrite (3.7) as

$$\mathcal{A} \;=\; \frac{1}{2}\Big\{\big(\boldsymbol{y} \wedge M^1\boldsymbol{x}\big) \wedge \big(\boldsymbol{y} \wedge M^2\boldsymbol{x}\big)\Big\},$$

which we can further rewrite in terms of the determinant of a matrix:

$$\mathcal{A} \;=\; \frac{1}{2}\cdot \det\,[\,M^1\boldsymbol{x} \mid M^2\boldsymbol{x} \mid \boldsymbol{y}\,]\,\boldsymbol{y}.$$

Accordingly, the condition that the alternating part of the LHS of (3.3) $= O$, reduces to

$$\det\,[\,M^1\boldsymbol{x} \mid M^2\boldsymbol{x} \mid \boldsymbol{y}\,] \;=\; 0, \tag{3.8}$$

since $\boldsymbol{y}$ is not a zero vector. By using notation $T_{ijk\ell}$, we can rewrite (3.8) as

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \varepsilon(i\,j\,\sigma(i,j)) \cdot T_{ijk\ell}\, x^k x^\ell\, y^{\sigma(i,j)} \;=\; 0, \tag{3.9}$$

where σ is a function defined by

$$\sigma(i,j) \;:=\; \{1,2,3\} - \{i,j\} \qquad (i \neq j;\; i,j \in \{1,2,3\});$$

for $i, j, k \in \{1,2,3\}$, ε is defined by

$$\varepsilon(i\,j\,k) \;:=\; \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123). \end{cases}$$

Note that in this case, in order to stress $i \neq j$, we dare to write the summation symbol with respect to indices i and j which are neither symmetrized nor alternized. We can see that the indices i and j in the LHS of (3.9) are alternating since exchanging i and j results in a reversal of the sign of the same term in $\boldsymbol{x}$ and $\boldsymbol{y}$. Whereas exchanging k and ℓ results in the same term; indices k and ℓ are symmetric to each other. Thus, we may alternize i and j, and symmetrize k and ℓ in the LHS of (3.9), which yields

$$\psi(i,j,k,\ell)\, T_{[i\,j](k\,\ell)}\, x^{(k}x^{\ell)}\, y^{\sigma(i,j)} \;=\; 0. \tag{3.10}$$

Here $[\cdots]$ implies the alternization of the indices there; $T_{[i\,j](k\,\ell)}$ is, therefore, defined by

$$T_{[i\,j](k\,\ell)} \;:=\; \frac{1}{2!\cdot 2!}\left(T_{ijk\ell} - T_{jik\ell} + T_{ij\ell k} - T_{ji\ell k}\right).$$

(3.10) is the constraint equation derived from $\mathcal{A} = O$. $T_{[i\,j](k\,\ell)}$ are independent unknowns whose number is 18; (3.10) is the linear homogeneous equation with $T_{[i\,j](k\,\ell)}$. Each point correspondence gives an equation in the form of (3.10). We thus can, in general, uniquely determine $T_{[i\,j](k\,\ell)}$ up to a scale factor with linear computation if we have 17 point correspondences. It is important to remark that the alternization of i and j, and the symmetrization of k and ℓ reduced the number of independent unknowns from 36 to 18 in this case.

Combining this with the result in the previous section, we obtain the following theorem.

Theorem 3.2 Let the tensor product $T_{ijk\ell}$ of two transformation matrices, M^1 and M^2, be decomposed into its symmetric part $\mathcal{S}$ and its alternating part $\mathcal{A}$. Then, $\mathcal{S}$ reduces to $T_{(i\,j)(k\,\ell)}$ that has 36 independent entries; $\mathcal{A}$ reduces to $T_{[i\,j](k\,\ell)}$ that has 18 independent entries. In addition, in general, we can uniquely determine $T_{(i\,j)(k\,\ell)}$ up to a scale factor with linear computation if we have 12 point correspondences (see (3.6)); whereas we need 17 point correspondences to uniquely determine $T_{[i\,j](k\,\ell)}$ up to a scale factor (see (3.10)). $\square$

It should be noted that the two scale factors incurred in determining $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$ are different in general.

Remark 3.1 (3.8) indicates that three vectors $M^1\boldsymbol{x}$, $M^2\boldsymbol{x}$ and $\boldsymbol{y}$ are coplanar. In other words, $\boldsymbol{y}$ cannot be any vector in $\mathcal{P}^2$. Hence, the alternating part imposes the "coplanarity condition" on $\boldsymbol{y}$ from a geometrical point of view. Whereas we may geometrically interpret the symmetric part as the "parallel condition" on the $\boldsymbol{y}$ coplanar with $M^1\boldsymbol{x}$ and $M^2\boldsymbol{x}$. This is because, if the concerned point is in plane 1, then $\boldsymbol{y}$ is parallel to $M^1\boldsymbol{x}$; if the point is in plane 2, then $\boldsymbol{y}$ is parallel to $M^2\boldsymbol{x}$. $\square$

3.4 Critical conditions in determining each part

So far, we have discussed the general case. In other words, we have assumed that we are given two planes such that they enable us to uniquely determine $T_{(ij)(k\ell)}$ and $T_{[ij](k\ell)}$ up to a scale factor, respectively. We have also assumed that we are given point correspondences such that they enable us to determine $T_{(ij)(k\ell)}$ and $T_{[ij](k\ell)}$. Here, we investigate the condition, which we call the *critical condition*, where we cannot uniquely determine $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$. Two cases may be distinguished. One is the case where two planes themselves do not allow us to determine $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$. The other is the case where points in a special configuration do not allow us to do so. We investigate each case separately in the subsequent sections.

3.4.1 Critical condition for two planes

To investigate the critical conditions for two planes, i.e., conditions such that two planes themselves do not allow us to uniquely determine $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$ up to a scale factor, respectively, we may assume that the points in each plane are distributed randomly. Namely, the points in each plane have no special configuration. Therefore, we may focus on any special relation among the normal vectors of the two planes and motion parameters, i.e., translation vectors and rotation matrices, of the two planes. If rotations of the two planes differ, we may expect that we can determine $T_{(ij)(k\ell)}$ and $T_{[ij](k\ell)}$ since different rotations imply the general case of two moving planes. Thus, we may concentrate on the case where two planes share the same rotation, and the normal vectors of the two planes and the translation vectors satisfy a special relation. A special relation could be considered in several ways; below, however, only "parallelism" and "perpendicularity" are considered. And we show that we can almost always determine $T_{(ij)(k\ell)}$; whereas a case exists where we cannot determine $T_{[ij](k\ell)}$.

When two planes share the same rotation, in order to analyze indeterminacy of $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$, we should first note that the rotation need not be considered, since the rotation only plays the role of changing the "representation" of points. In other words, the orientation of the optical axis does not play any role. Hence, it suffices to assume that the two planes are translated with $R = I$. We recall that a transformation matrix M is expressed by

$$M = I + \boldsymbol{t}\boldsymbol{n}^{\mathrm{T}}$$

in this case, and $T_{(ij)(k\ell)}$ is derived from the symmetric part of the tensor product of two transformation matrices; while $T_{[i\,j](k\,\ell)}$ is derived from the alternating part. As we can see, for a plane, the normal vector $\boldsymbol{n}$ and the translation vector $\boldsymbol{t}$ play an equivalent role in $T_{(i\,j)(k\,\ell)}$; whereas they do not in $T_{[i\,j](k\,\ell)}$. Remember that we have four vectors in $\mathbf{R}^3$: two are normal vectors and the other two are translation vectors. We denote by $\boldsymbol{n}^i$ the normal vector of plane i $(i = 1, 2)$ and by $\boldsymbol{t}^i$ the translation vector of plane i $(i = 1, 2)$.

We first discuss the case of $T_{(i\,j)(k\,\ell)}$. Imposing parallelism or perpendicularity on any two (or more) of $\boldsymbol{n}^1$, $\boldsymbol{n}^2$, $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$ does not matter in determining $T_{(i\,j)(k\,\ell)}$. This is because we obtain the symmetric part by adding, to the tensor product of two transformation matrices, its transportation; this addition leads to no serious cause in determining $T_{(i\,j)(k\,\ell)}$. (This is not the case with $T_{[i\,j](k\,\ell)}$, as we will show shortly.) Therefore, we can (almost) always determine $T_{(i\,j)(k\,\ell)}$. It should be noted that we cannot deny the possibility of the existence of a special relation different from parallelism or perpendicularity where we cannot determine $T_{(i\,j)(k\,\ell)}$, but concerning such a special relation is beyond the scope of this paper.

We now turn to the case of $T_{[i\,j](k\,\ell)}$. We obtain the alternating part by subtracting from the tensor product, its transportation. This subtraction causes indeterminacy of $T_{[i\,j](k\,\ell)}$ when $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ are parallel or $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$ are parallel. The following theorem states that for two planes 1 and 2 such that $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ are parallel, we cannot determine $T_{[i\,j](k\,\ell)}$. It also states that for two planes moving in the same direction (namely, $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$ are parallel), we cannot determine $T_{[i\,j](k\,\ell)}$. Accordingly, now that all the constraint equations are homogeneous in $T_{[i\,j](k\,\ell)}$, if we observe that the number of zero-eigenvalues of the coefficient matrix to determine $T_{[i\,j](k\,\ell)}$ is not three, then we can completely deny two cases: one is the case where the observed two planes are translated to the same direction with the same rotation; the other is the case where the normal vectors of the two planes are parallel. We note that "the number of zero-eigenvalues is three" does not always imply the two cases above. This is because the theorem below ensures only the sufficient condition; it does not ensure the necessary condition.

Theorem 3.3 Suppose that two planes 1 and 2 share the same rotation. Also suppose that the correspondences of two images (before and after the motions) of P $(P \geq 17)$ general points in plane 1 or 2, are known. Let A be the coefficient matrix of $P \times 18$ to determine $T_{[i\,j](k\,\ell)}$ (see (3.10)). If (i) $\boldsymbol{t}^2 = \alpha\,\boldsymbol{t}^1$ or (ii) $\boldsymbol{n}^2 = \beta\,\boldsymbol{n}^1$, then

$$\operatorname{rank} A \;=\; 15.$$

In particular, when both (i) and (ii) hold, and $\alpha \cdot \beta = 1$, then $\operatorname{rank} A = 10$.

Proof: The proof of this theorem will be given in Appendix A. $\square$

3.4.2 Critical condition for point configuration

In contrast to the above section, even though two planes allow us to determine $T_{(ij)(k\ell)}$ and $T_{[ij](k\ell)}$, we may have a case where the choice of points leads to indeterminacy of $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$. Here we are interested in the case where points in a special configuration cause indeterminacy of $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$. In this section, we assume that two planes themselves allow us to uniquely determine $T_{(ij)(k\ell)}$ and $T_{[ij](k\ell)}$ up to a scale factor, respectively.

Indeterminacy of $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$ is equivalent to the existence of at least one spurious solution of (3.6) or (3.10). Therefore, we obtain a critical condition from the properties we have under the assumption that we have no spurious solution of (3.6) or (3.10). Below, we only discuss the number of points we need in each plane. Points in another configuration that cause indeterminacy of $T_{(ij)(k\ell)}$ or $T_{[ij](k\ell)}$ are not considered in this paper.

The theorem below states that unless we have at least four points in each plane such that no three of them are collinear, we cannot determine $T_{(ij)(k\ell)}$. It also states that if we do not have four points in a plane, then the number of spurious solutions increases by two every time the number of points in the plane decreases by one.

On the other hand, we give a conjecture below with respect to $T_{[ij](k\ell)}$. (We obtained this conjecture through our simulation results (cf. Fig. 2); we have not proved it.) It implies that unless we have at least seven points in each plane or have four points in each plane such that no three of them are collinear, we cannot determine $T_{(ij)(k\ell)}$. It also implies that if we do not have seven points in a plane, then the number of spurious solutions increases by one every time the number of points in the plane decreases by one.

Theorem 3.4

(1) Let P_1 and P_2 be the number of points in planes 1 and 2, respectively. If we uniquely determine $T_{(ij)(k\ell)}$ up to a scale factor, then we have

$$\begin{aligned} P_1 + P_2 &\geq 12, \\ \min(P_1, P_2) &\geq 4, \end{aligned}$$

and four points in each plane such that no three of them are collinear.

(2) Suppose $P_1 + P_2 \geq 12$ and $P_1 \leq 3$. We suppose that we have four points in plane 2 such that no three of them are collinear. We also suppose that three points in plane 1 are not collinear if $P_1 = 3$. Let S be the coefficient matrix of $3(P_1 + P_2) \times 36$ to determine $T_{(ij)(k\ell)}$ (see (3.6)). Then, we have

$$\dim(\mathrm{Ker}\, S) \;=\; 9 - 2P_1,$$

where Ker$S$ denotes the kernel of matrix S.

Proof: The proof of this theorem will be given in Appendix B. □

Conjecture 3.1

(1) If we can uniquely determine $T_{[ij](k\ell)}$ up to a scale factor, then we have

$$\begin{aligned} P_1 + P_2 &\geq 17, \\ \min(P_1,\, P_2) &\geq 7, \end{aligned}$$

and four points in each plane such that no three of them are collinear.

(2) We suppose $P_1 + P_2 \geq 17$ and $P_1 \leq 6$. We also suppose that we have four points in plane 2 such that no three of them are collinear. Then, we have

$$\dim(\mathrm{Ker}\, A) \;=\; 8 - P_1,$$

where A is the coefficient matrix of $(P_1 + P_2) \times 18$ to determine $T_{[ij](k\ell)}$ (see (3.10)). □

Remark 3.2 $P_1 = 0$ indicates that all observed points exist in only plane 2; this is equivalent to the case where one plane alone exists. In this case, we have $\dim(\mathrm{Ker}\, S) = 9$ and $\dim(\mathrm{Ker}\, A) = 8$. □

We suppose that $P_1 \leq 6$ and $P_1 + P_2 \geq 17$. Combining Theorem 3.4 and Conjecture 3.1, we can estimate the possible number of points in plane 1 just by counting the dimensions of Ker$S$ and Ker$A$. Note that dim(Ker$S$) and dim(Ker$A$) coincide with the number of zero-eigenvalues of S and A, respectively, since S and A are both coefficient matrices of linear homogeneous systems. Let s and a denote the number of zero-eigenvalues of S and that of A, respectively;

we use notation (s, a). Table 1 shows the possible number of P_1, depending on the number of zero-eigenvalues of S and A. We remark that this table shows just a possibility. It may be possible to estimate another number of P_1 based on other critical conditions (if any) not concerned in this paper.

4 Decomposition into two transformation matrices

In Section 3, we showed that if we have at least 17 point correspondences, we can generally determine $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$. Here we discuss how we obtain two transformation matrices from computed $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$. We recall that $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$ have an unknown scale factor, respectively, and that the two scale factors are different from each other. Let $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$ be computed as $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$, respectively. Then we have

$$\begin{aligned}\hat{T}_{(i\,j)(k\,\ell)} &= \rho\, T_{(i\,j)(k\,\ell)} \qquad (\rho \in \mathbf{R}^*),\\ \hat{T}_{[i\,j](k\,\ell)} &= \tau\, T_{[i\,j](k\,\ell)} \qquad (\tau \in \mathbf{R}^*).\end{aligned}$$

We should first note that an arbitrary vector in $\mathbf{R}^{36}$ and that in $\mathbf{R}^{18}$ together are not always decomposed into two 3×3 matrices. We say two vectors, one in $\mathbf{R}^{36}$ and the other in $\mathbf{R}^{18}$, are *decomposable* if they are decomposed into two 3×3 matrices. To see whether or not $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$ are both decomposable, we only have to verify the following equations for all i, j, k, ℓ and their primes:

$$\hat{T}_{(i\,j)(k\,\ell)} \cdot \hat{T}_{(i'\,j')(k'\,\ell')} = \hat{T}_{(i'\,j)(k'\,\ell)} \cdot \hat{T}_{(i\,j')(k\,\ell')}, \tag{4.1}$$

$$\hat{T}_{[i\,j](k\,\ell)} \cdot \hat{T}_{[i'\,j'](k'\,\ell')} = \hat{T}_{[i'\,j](k'\,\ell)} \cdot \hat{T}_{[i\,j'](k\,\ell')}, \tag{4.2}$$

where $i, j, k, \ell, i', j', k', \ell' \in \{1, 2, 3\}$. (4.1) and (4.2) are the immediate results of the fact that $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$ are both tensors of degree 2. The two equations are both quadratic in $\hat{T}_{(i\,j)(k\,\ell)}$ or $\hat{T}_{[i\,j](k\,\ell)}$. Below, we assume that both (4.1) and (4.2) are satisfied. We then consider how to decompose $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$ together into two transformation matrices M^1 and M^2. We remark that each of M^1 and M^2 has a scale factor; we cannot determine the two scale factors due to homogeneity.

The decomposition procedure is divided into two main steps: determining every column vector of M^1 or M^2 with an unknown scale factor; and determining the scale factors there to uniquely recover M^1 and M^2 up to a scale factor, respectively. In preparation for further investigation, we define 3×3 matrices, which together form a partition of $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$:

$$U_1 := \left(\hat{T}_{(i\,i)(k\,k)}\right), \qquad U_3 := \left(\hat{T}_{(i\,j)(k\,k)}\right),$$
$$U_2 := \left(\hat{T}_{(i\,i)(k\,\ell)}\right), \qquad U_4 := \left(\hat{T}_{(i\,j)(k\,\ell)}\right);$$
$$V_3 := \left(\hat{T}_{[i\,j](k\,k)}\right), \qquad V_4 := \left(\hat{T}_{[i\,j](k\,\ell)}\right),$$

where i and j together are indices for the column-numbers, and k and ℓ together are for the row-numbers. In the first step, we use U_1, U_3 and V_3; in the second step, we use the other three. In the subsequent sections, for simplicity, we assume that no entry of M^1 and M^2 is zero. Note that when some of them are zero, based on the procedure below, we can devise a similar procedure to obtain M^1 and M^2.

Remark 4.1 M^1 and M^2 together have 18 entries whereas we have 54 $(= 36 + 18)$ quadratic constraint equations in them. This implies that we need not necessarily use all the constraint equations to determine them; instead, we may solve a simultaneous quadratic equation system with 18 unknowns. However, solving such a system is not an easy computational task and, furthermore, we may have a number of spurious solutions (theoretically, we may have 2^{18} solutions). Such a method is far from practical. In contrast to this, our decomposition procedure below uses all the constraint equations. This makes the procedure computationally simple well as robust with respect to noise. It also gives a unique solution. □

4.1 Determining column vectors

From the definition, we can see that

$$\begin{aligned} U_1 &= \left(\rho\, M^1_{ik}\, M^2_{ik}\right), \\ U_3 &= \left(\rho\, \frac{M^1_{ik}\, M^2_{jk} + M^1_{jk}\, M^2_{ik}}{2}\right), \\ V_3 &= \left(\tau\, \frac{M^1_{ik}\, M^2_{jk} - M^1_{jk}\, M^2_{ik}}{2}\right). \end{aligned}$$

For two matrices N_1 and N_2 of the same size, we define N_1/N_2 as a matrix of that size such that an entry of N_1/N_2 is obtained by dividing its correspondent entry of N_1 by that of N_2. Putting

$$W^1 := \frac{2U_3}{U_1}, \qquad W^2 := \frac{2V_3}{U_1},$$

we have, for $k \in \{1,2,3\}$,

$$\text{the } k\text{-th row of } W^1 = \left(\frac{M^2_{2k}}{M^2_{1k}} + \frac{M^1_{2k}}{M^1_{1k}}, \frac{M^2_{3k}}{M^2_{2k}} + \frac{M^1_{3k}}{M^1_{2k}}, \frac{M^2_{1k}}{M^2_{3k}} + \frac{M^1_{1k}}{M^1_{3k}} \right),$$
$$\text{the } k\text{-th row of } W^2 = \frac{1}{\varphi}\left(\frac{M^2_{2k}}{M^2_{1k}} - \frac{M^1_{2k}}{M^1_{1k}}, \frac{M^2_{3k}}{M^2_{2k}} - \frac{M^1_{3k}}{M^1_{2k}}, \frac{M^2_{1k}}{M^2_{3k}} - \frac{M^1_{1k}}{M^1_{3k}} \right),$$

where $\varphi = \frac{\rho}{\tau}$. Moreover, by defining

$$r^1_{2k} := \frac{M^1_{2k}}{M^1_{1k}}, \qquad \tilde{r}^1_{3k} := \frac{M^1_{1k}}{M^1_{3k}}; \tag{4.3}$$
$$r^2_{2k} := \frac{M^2_{2k}}{M^2_{1k}}, \qquad \tilde{r}^2_{3k} := \frac{M^2_{1k}}{M^2_{3k}}, \tag{4.4}$$

we obtain the following simultaneous algebraic equations with five unknowns:

$$\left\{ \begin{array}{rcl} r^2_{2k} + r^1_{2k} & = & W^1_{k1} \\ \tilde{r}^2_{3k} + \tilde{r}^1_{3k} & = & W^1_{k3} \\ \dfrac{1}{r^2_{2k} \cdot \tilde{r}^2_{3k}} + \dfrac{1}{r^1_{2k} \cdot \tilde{r}^1_{3k}} & = & W^1_{k2} \\ r^2_{2k} - r^1_{2k} & = & \varphi W^2_{k1} \\ \tilde{r}^2_{3k} - \tilde{r}^1_{3k} & = & \varphi W^2_{k3} \\ \dfrac{1}{r^2_{2k} \cdot \tilde{r}^2_{3k}} - \dfrac{1}{r^1_{2k} \cdot \tilde{r}^1_{3k}} & = & \varphi W^2_{k2}. \end{array} \right. \tag{4.5}$$

We have exactly two solutions of (4.5), and the two solutions are essentially equivalent. This can be understood in the following way. If $(r^1_{2k}, \tilde{r}^1_{3k}, r^2_{2k}, \tilde{r}^2_{3k}, \varphi)$ is a solution of (4.5), then $(r^2_{2k}, \tilde{r}^2_{3k}, r^1_{2k}, \tilde{r}^1_{3k}, -\varphi)$ is also a solution. Hence, we have at least two solutions (it is easy to see that the two are essentially equivalent). On the other hand, the first five equations are reduced to an equation of degree 4 in φ and the other four unknowns are linearly related to φ. We thus have at most four solutions. Two of the four are found to be spurious by the sixth equation. This is because (4.5) gives six algebraically independent equations in $r^1_{2k}, \tilde{r}^1_{3k}, r^2_{2k}, \tilde{r}^2_{3k}$ and φ. Accordingly, the number of solutions is exactly two; they are essentially equivalent. Note that the case where the four solutions are all found to be spurious, implies that we cannot decompose $\hat{T}_{(ij)(k\ell)}$ and $\hat{T}_{[ij](k\ell)}$ together into M^1 and M^2; this contradicts our assumption. It is important to remark that we easily obtain the two solutions of (4.5) since we can symbolically solve the first five equations in (4.5).

For each k ($\in \{1,2,3\}$), we can uniquely determine the k-th column vector of M^1 up to a scale factor from r^1_{2k} and $\tilde{r}^1_{3k}$; whereas we determine the k-th column vector of M^2 from r^2_{2k} and $\tilde{r}^2_{3k}$. Note that the column vectors of M^1 are independently determined; we have eight combinations in constructing M^1 from the column vectors. This should also be applied to M^2. However, φ enables us to uniquely construct M^1 and M^2 from the column vectors. This is because the value of φ is independent of k (i.e., for any $k \in \{1,2,3\}$, the same value of φ is obtained by solving (4.5)) and because only the same value of φ is permitted in M^1 and M^2. Therefore, we determine every column vector of M^1 or M^2 with an unknown scale factor:

$$M^1 = [\gamma^1_1 \boldsymbol{m}^1_1 \mid \gamma^1_2 \boldsymbol{m}^1_2 \mid \gamma^1_3 \boldsymbol{m}^1_3], \tag{4.6}$$

$$M^2 = [\gamma^2_1 \boldsymbol{m}^2_1 \mid \gamma^2_2 \boldsymbol{m}^2_2 \mid \gamma^2_3 \boldsymbol{m}^2_3], \tag{4.7}$$

where $\boldsymbol{m}^1_i$, $\boldsymbol{m}^2_i$ are known vectors and γ^1_i, γ^2_i are unknown ($i = 1,2,3$).

In the next section, we determine scale factors γ^1_i and γ^2_i by using U_2, U_4 and V_4; we uniquely determine M^1 and M^2 up to a scale factor, respectively. Note that φ is already determined; the case where the sign of φ is reversed reduces to the same results.

4.2 Determining the scale factors

Defining $W^3 := U_2$ and $W^4 := U_4 + \varphi V_4$, we see that

$$W^3 = \left(\rho \frac{M^1_{ik} M^2_{i\ell} + M^1_{i\ell} M^2_{ik}}{2} \right), \tag{4.8}$$

$$W^4 = \left(\rho \frac{M^1_{ik} M^2_{j\ell} + M^1_{i\ell} M^2_{jk}}{2} \right). \tag{4.9}$$

(4.6) and (4.7) respectively allow us to express M^1_{ik} and $M^2_{i\ell}$ as

$$M^1_{ik} = \gamma^1_k (\boldsymbol{m}^1_k)_i,$$

$$M^2_{i\ell} = \gamma^2_\ell (\boldsymbol{m}^2_\ell)_i,$$

where $(\boldsymbol{m}^1_k)_i$ denotes the i-th component of $\boldsymbol{m}^1_k$ and $(\boldsymbol{m}^2_k)_\ell$ the ℓ-th of $\boldsymbol{m}^2_k$. Substituting these into the $(k\,\ell)$-th row ($(k\,\ell) = (2\,3), (3\,1), (1\,2)$) of (4.8) and (4.9), we then obtain simultaneous linear equations with two unknowns $\rho\,\gamma^1_k\,\gamma^2_\ell$ and $\rho\,\gamma^1_\ell\,\gamma^2_k$:

$$\begin{cases} \dfrac{(\boldsymbol{m}^1_k)_i\,(\boldsymbol{m}^2_\ell)_i}{2}\,\rho\,\gamma^1_k\,\gamma^2_\ell + \dfrac{(\boldsymbol{m}^1_\ell)_i\,(\boldsymbol{m}^2_k)_i}{2}\,\rho\,\gamma^1_\ell\,\gamma^2_k = W^3_{(k\,\ell),(i\,i)} \\ \dfrac{(\boldsymbol{m}^1_k)_i\,(\boldsymbol{m}^2_\ell)_j}{2}\,\rho\,\gamma^1_k\,\gamma^2_\ell + \dfrac{(\boldsymbol{m}^1_\ell)_i\,(\boldsymbol{m}^2_k)_j}{2}\,\rho\,\gamma^1_\ell\,\gamma^2_k = W^4_{(k\,\ell),(i\,j)}, \end{cases} \tag{4.10}$$

where $(i\,i) = (1\,1), (2\,2), (3\,3)$ in the top equation; $(i\,j) = (2\,3), (3\,1), (1\,2)$ in the bottom equation. (4.10) gives six equations; it is an overdetermined system of linear equations in $\rho\gamma_k^1\gamma_\ell^2$ and $\rho\gamma_\ell^1\gamma_k^2$. Note that at least two equations in (4.10) are independent. No solution in this system implies that we cannot decompose $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$ into M^1 and M^2; this contradicts our assumption. Hence, the six equations in (4.10) reduce to only two independent ones. Accordingly, we can determine $\rho\gamma_k^1\gamma_\ell^2$ and $\rho\gamma_\ell^1\gamma_k^2$ by solving (4.10).

We now have the values of $\rho\gamma_k^1\gamma_\ell^2$ and $\rho\gamma_\ell^1\gamma_k^2$ ($(k\,\ell) = (2\,3), (3\,1), (1\,2)$). Taking ratios of them, we obtain $\gamma_1^1 : \gamma_2^1 : \gamma_3^1$ and $\gamma_1^2 : \gamma_2^2 : \gamma_3^2$, which yield unique M^1 and M^2 up to a scale factor, respectively.

Remark 4.2 From a theoretical point of view, two of the six equations in (4.10) are sufficient to obtain $\rho\gamma_k^1\gamma_\ell^2$ and $\rho\gamma_\ell^1\gamma_k^2$, and the others are redundant; whereas from a practical point of view, it is better to use all six equations via the method of least squares to obtain the solution. Using all of the equations leads to robustness of the solution with respect to noise. □

5 Description of algorithm

In Section 3, we showed that if we have at least 17 point correspondences, we can generally determine $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$, and that they are independently determined as the solution of a linear homogeneous system, respectively (see (3.6) and (3.10)). $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$ are uniquely determined up to a scale factor, respectively. In Section 4, we showed how to decompose $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$ together into transformation matrices M^1 and M^2 where each column vector of M^1 and M^2 is first determined up to a scale factor, and the scale factors there are then determined to uniquely recover M^1 and M^2 up to a scale factor, respectively. Note that φ plays an important role in constructing M^1 and M^2 from the column vectors. Based on these results, here we describe an algorithm for computing transformation matrices from given point correspondences.

We assume that we are given the coordinates of two images (before and after a motion) of a point in one of two planes. We also assume that the given point correspondences allow us to uniquely determine $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$ up to a scale factor, respectively. If the given point correspondences do not allow us to determine $T_{(i\,j)(k\,\ell)}$ and $T_{[i\,j](k\,\ell)}$, we then may shift our attention to the results in Section 3.4; we estimate a configuration of given points or a relation among the translation vectors and the normal vectors of two planes.

The following algorithm is obtained for computing M^1 and M^2 from given point correspondences $\{\boldsymbol{x}_p, \boldsymbol{y}_p\}_{p=1}^{P}$. We recall that once a transformation matrix is known, we can (almost uniquely) determine [18] the rotation matrix, the translation vector and the normal vector of the plane, where singular value decomposition of the transformation matrix plays the central role. We should note that $P \geq 17$ and at least seven points exist in each plane. We should also note that $i, j, k, \ell \in \{1, 2, 3\}$.

Step 1 [Determining the symmetric part and the alternating part]

(1): Create a $3P \times 36$ matrix S such that the $(q, (i\,j)(k\,\ell))$-component $S_{p,(i\,j)(k\,\ell)}$ is

$$\begin{cases} S_{3p-2,(i\,j)(k\,\ell)} & := \psi(i,j,k,\ell)\,[\,\boldsymbol{y}_p\,]^{1(i}\,[\,\boldsymbol{y}_p\,]^{|1|j)}\,x_p^{(k}\,x_p^{\ell)} \\ S_{3p-1,(i\,j)(k\,\ell)} & := \psi(i,j,k,\ell)\,[\,\boldsymbol{y}_p\,]^{2(i}\,[\,\boldsymbol{y}_p\,]^{|2|j)}\,x_p^{(k}\,x_p^{\ell)} \\ S_{3p,(i\,j)(k\,\ell)} & := \psi(i,j,k,\ell)\,[\,\boldsymbol{y}_p\,]^{1(i}\,[\,\boldsymbol{y}_p\,]^{|2|j)}\,x_p^{(k}\,x_p^{\ell)} \end{cases} \qquad (p = 1, 2, \ldots, P),$$

where $\boldsymbol{x}_p = (x_p^1, x_p^2, x_p^3)^{\mathrm{T}}$. Also create a $P \times 18$ matrix A such that the $(p, [i\,j](k\,\ell))$-component $A_{p,[i\,j](k\,\ell)}$ is

$$A_{p,[i\,j](k\,\ell)} := \psi(i,j,k,\ell)\,x_p^{(k}\,x_p^{\ell)}\,y_p^{\sigma(i,j)} \quad (p = 1, 2, \ldots, P).$$

(2): Compute an eigenvector that corresponds to the zero-eigenvalue of $S^{\mathrm{T}}S$; let $\hat{T}_{(i\,j)(k\,\ell)}$ be the eigenvector. Also compute an eigenvector that corresponds to the zero-eigenvalue of $A^{\mathrm{T}}A$; let $\hat{T}_{[i\,j](k\,\ell)}$ be the eigenvector. (QR decomposition may be useful.)

Step 2

Create 3×3 matrices U_1, U_2, U_3, U_4, V_3 and V_4 from $\hat{T}_{(i\,j)(k\,\ell)}$ and $\hat{T}_{[i\,j](k\,\ell)}$.

Step 3 [Determining each column vector]

(1): $W^1 := \dfrac{2U_3}{U_1}$;

$W^2 := \dfrac{2V_3}{U_1}$.

(2): For each k $(\in \{1, 2, 3\})$, solve (4.5); let $(r_{2k}^1,\, \tilde{r}_{3k}^1,\, r_{2k}^2,\, \tilde{r}_{3k}^2,\, \varphi)$ be a solution (we have two solutions which are essentially equivalent).

(3) : For k ($\in \{1,2,3\}$),

$$m_k^1 := \left(1, r_{2k}^1, \frac{1}{\tilde{r}_{3k}^1}\right)^{\mathrm{T}};$$

$$m_k^2 := \left(1, r_{2k}^2, \frac{1}{\tilde{r}_{3k}^2}\right)^{\mathrm{T}}.$$

(φ should be the same here and the φ should be used in the next step.)

Step 4 [Determining the scale factors]

(1) : $W^3 := U_2$;

$W^4 := U_4 + \varphi V_4$.

(2) : For each $(k\,\ell)$ ($\in \{(2\,3),(3\,1),(1\,2)\}$), solve (4.10); let $\rho\gamma_k^1\gamma_\ell^2$ and $\rho\gamma_\ell^1\gamma_k^2$ be the solution. (The method of least squares may be useful.)

(3) : Compute $\gamma_1^1 : \gamma_2^1 : \gamma_3^1$ and $\gamma_1^2 : \gamma_2^2 : \gamma_3^2$.

(4) : Construct M^1 and M^2 (cf. (4.6), (4.7)). □

6 Conclusion

We have investigated the structure-from-motion problem under the conditions that two planes independently and rigidly move and that, for given two perspective images, the correspondences of points in the planes are known.

We showed that we can generally determine the normal vectors of the two planes and the two motions when we observe 17 points, where the tensor product of two transformation matrices and its decomposition play the central role. Our procedure for determining them is divided into two parts: one is to determine the symmetric part and the alternating part of the tensor product, and the other is to decompose the two parts together into the two transformation matrices. Note that singular value decomposition of a transformation matrix leads to determining (almost uniquely) the normal vector of a plane and the motion parameters.

In the first part, we decomposed the tensor product of the two transformation matrices into its symmetric part and its alternating part, and investigated each part separately where the symmetrization and the alternization enabled us to reduce the number of unknowns. The symmetric part has 36 independent unknowns and each point correspondence gives three linear homogeneous constraint equations to them. In contrast, the alternating part has 18 independent unknowns and each point correspondence gives one linear homogeneous constraint equation to them. Therefore, in general, we can uniquely determine each part up to a scale factor with linear computation. In addition, we geometrically characterized the constraint on the alternating part as a "coplanarity condition" on the coordinates in the second image, and the constraints on the symmetric part as a "parallel condition".

In the second part, we gave a procedure to decompose the two parts together into two transformation matrices. The procedure makes full use of the constraint equations in order to determine two transformation matrices from the two parts. Using all of the constraint equations leads to a computationally simple procedure as well as a robust one with respect to noise. It also gives a unique solution up to a scale factor (eliminating the scale factor is impossible due to homogeneity). Our decomposition procedure has two main steps: one is determining every column vector of the transformation matrices with an unknown scale factor, and the other is determining the scale factors there to uniquely recover the two transformation matrices up to a scale factor, respectively. Here the ratio of the scale factors incurred in determining the symmetric part and the alternating part plays an important role in uniquely constructing the transformation matrices.

We also investigated some critical conditions, namely, conditions that do not allow us to determine the symmetric part or the alternating part. Two cases were distinguished: one is the case where two planes themselves do not allow us to determine the symmetric part or the alternating one, and the other is the case where points in a special configuration do not allow us to do so. In the first case, we showed that when two planes share the same rotation, we cannot determine the alternating part if the two normal vectors are parallel or the two translation vectors are parallel. In the second case, we showed that unless we have at least four points in each plane such that no three of them are collinear, we cannot determine the symmetric part. Whereas we gave a conjecture for the alternating part: unless we have at least seven points in each plane or have four points in each plane such that no three of them are collinear, we cannot determine the alternating part. Furthermore, we showed that when the number of points in one of two planes does not reach the critical number (4 for the symmetric part; 7 for the alternating part), we can estimate the possible number of points in the plane just by counting the number of zero-eigenvalues of the coefficient matrices to determine the two parts. We should remark that we have not investigated all critical conditions. In other words, in the first case we focused only on parallelism and perpendicularity, and in the second case we discussed only the number of points we need in each plane. Some other conditions (if any) may cause indeterminacy in the symmetric part or the alternating part. Investigation of other critical conditions is left open in this paper.

Throughout this paper, we also assumed that we are given the coordinates of two images (before and after a motion) of a point in one of two planes. This assumption is crucial in a certain sense because in practical situations, we have no way of knowing whether or not given points exist in one of two planes. As we have seen, when we determine the symmetric part and the alternating part (see (4.1) and (4.2)), we can determine whether or not a given set of point coordinates was obtained from two planes. However, this is not an easy computational task; we have a number of combinations. Instead, during the decomposition procedure, we may make use of the redundancy of the number of constraint equations in order to check it. Theoretical considerations on this problem are left open for future research.

Implementation details and practical efficiency of the proposed algorithm will be reported later.

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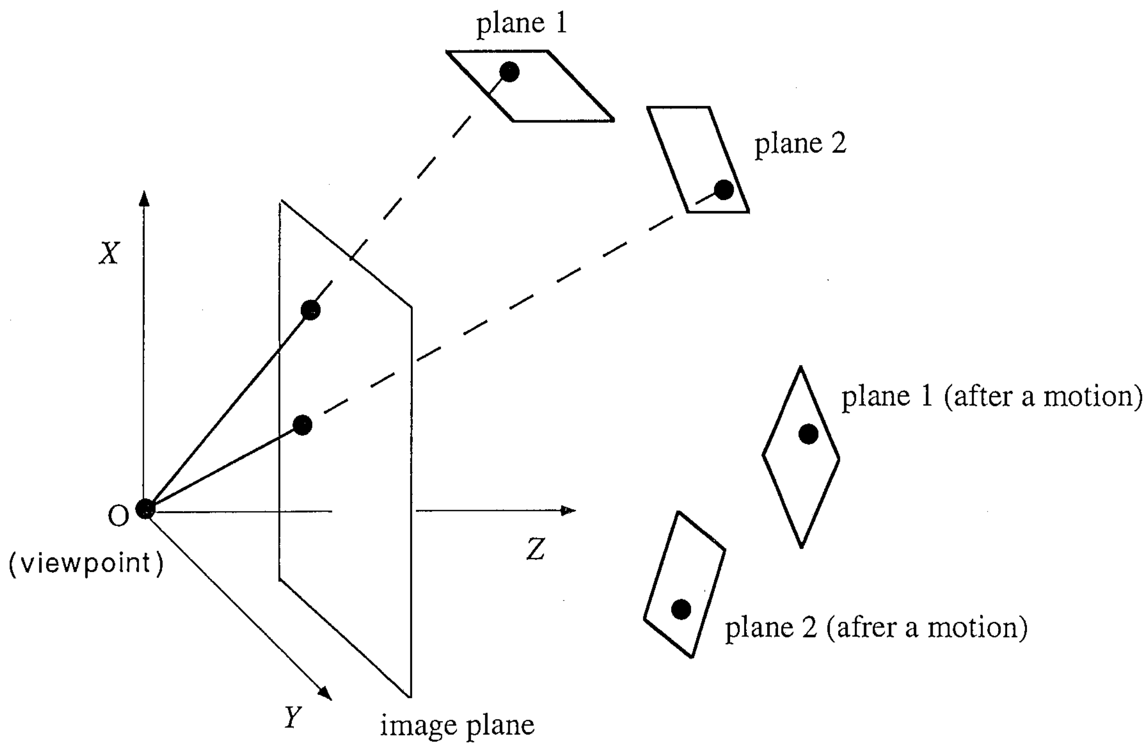

Fig. 1: Two independently and rigidly moving planes

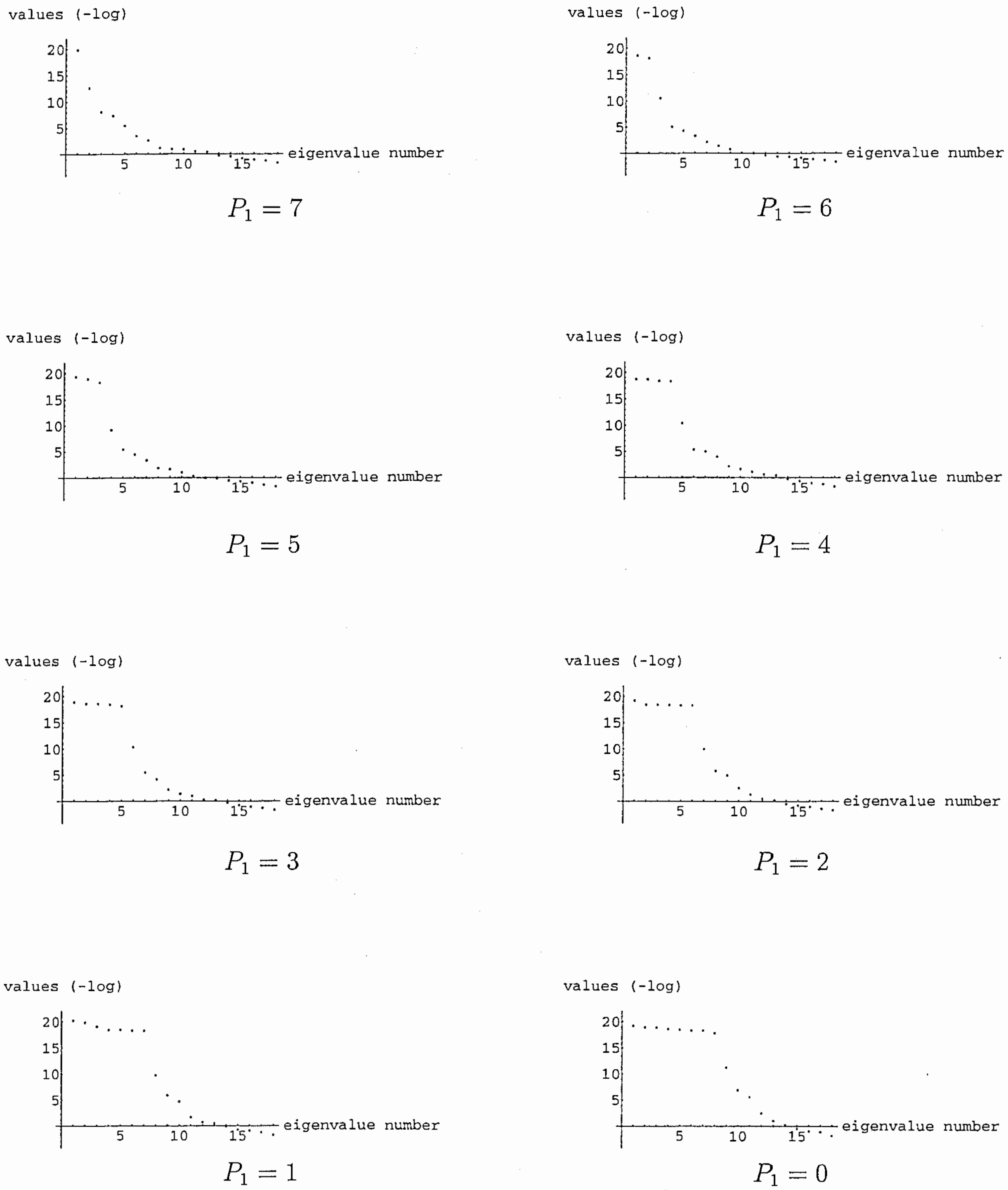

Fig. 2: P_1 and eigenvalues of A ($P_2 = 17 - P_1$; two planes and two motions were randomly chosen and fixed, points were randomly generated at each time)

Table 1: Estimation of possible number of P_1

$P_1+P_2 \geq 17$ and $P_1 \leq 6$ are assumed where P_i denotes the number of points in plane i $(i = 1, 2)$. s and a denote the number of zero-eigenvalues of the coefficient matrices in determining the symmetric part and the alternating part, respectively.

(s, a)	P_1
$(1, 2)$	6
$(1, 3)$	6 or 5
$(1, 4)$	4
$(3, 5)$	3
$(5, 6)$	2
$(7, 7)$	1
$(9, 8)$	0

A Proof of Theorem 3.3

Let $\boldsymbol{z}$ be a vector in $\mathbf{R}^{18}$ that is obtained by aligning the entries of $T_{[i\,j](k,\,\ell)}$. Here we align the entries in the order such that $[i\,j](k\,\ell) = [2\,3](1\,1), [3\,1](1\,1), [1\,2](1\,1), [2\,3](2\,2), \ldots, [1\,2](1\,2)$. We can then rewrite (3.10) in the form of a linear homogeneous system (in entries of $\boldsymbol{z}$) as follows:

$$A\,\boldsymbol{z} \;=\; 0. \tag{A.1}$$

Since A is a $P \times 18$ matrix and $P \geq 17$, $\operatorname{rank} A = 15$ is equivalent to $\dim(\operatorname{Ker} A) = 3$. As we can easily see, to prove $\dim(\operatorname{Ker} A) = 3$, it suffices to show (I) and (II) below.

(I) We have three linearly independent nontrivial solutions of (A.1).

(II) Any other solution of (A.1) is expressed as a linear combination of the three solutions.

Since $R = I$, we have

$$\begin{aligned} M^1 &= I + \boldsymbol{t}^1\,(\boldsymbol{n}^1)^{\mathrm{T}}, \\ M^2 &= I + \boldsymbol{t}^2\,(\boldsymbol{n}^2)^{\mathrm{T}}. \end{aligned}$$

$T_{[i\,j](k\,\ell)}$ is, therefore, reduced to

$$\begin{aligned} T_{[i\,j](k\,\ell)} \;=\; & t^1_{[i}\,t^2_{j]}\,n^1_{(k}\,n^2_{\ell)} + \frac{1}{4}\Big[\delta_{ik}\,(t^2_j\,n^2_\ell - t^1_j\,n^1_\ell) + \delta_{i\ell}\,(t^2_j\,n^2_k - t^1_j\,n^1_k) \\ & -\delta_{jk}\,(t^2_i\,n^2_\ell - t^1_i\,n^1_\ell) - \delta_{j\ell}\,(t^2_i\,n^2_k - t^1_i\,n^1_k)\Big], \end{aligned} \tag{A.2}$$

where δ_{ik} denotes the Kronecker delta ($\delta_{ik} = 1$ for $i = k$ and $= 0$ otherwise). This is the $[i\,j](k\,\ell)$-component of a solution of (A.1). For two cases, (i) $\boldsymbol{t}^2 = \alpha\,\boldsymbol{t}^1$ and (ii) $\boldsymbol{n}^2 = \beta\,\boldsymbol{n}^1$, we show (I) and (II).

Case i ($\boldsymbol{t}^2 = \alpha\,\boldsymbol{t}^1$) : Since $\boldsymbol{t}^2 = \alpha\,\boldsymbol{t}^1$, we can hide $\boldsymbol{t}^2$ in (A.2). The $[i\,j](k\,\ell)$-component of a solution is then rewritten as

$$\begin{aligned} T_{[i\,j](k\,\ell)} \;=\; & \frac{1}{4}\Big[\Big\{\delta_{ik}(\alpha n^2_\ell - n^1_\ell) + \delta_{i\ell}(\alpha n^2_k - n^1_k)\Big\}\,t^1_j \\ & - \Big\{\delta_{jk}(\alpha n^2_\ell - n^1_\ell) + \delta_{j\ell}(\alpha n^2_k - n^1_k)\Big\}\,t^1_i\Big]. \end{aligned}$$

Here $\boldsymbol{t}^i = (t^i_1, t^i_2, t^i_3)^{\mathrm{T}}$ and $\boldsymbol{n}^i = (n^i_1, n^i_2, n^i_3)^{\mathrm{T}}$. $R = I$ indicates that $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{t}^1$ if a point is in plane 1; $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{t}^2$ if it is in plane 2. In either case, we have

$$2\,t^1_3\,x^1x^1y^2 - 2\,t^1_2\,x^1x^1y^3 - 4\,\frac{t^1_3}{2}\,x^1x^2y^1 + 4\,\frac{t^1_1}{2}\,x^1x^2y^3 + 4\,\frac{t^1_2}{2}\,x^1x^3y^1 - 4\,\frac{t^1_1}{2}\,x^1x^3y^2 \;=\; 0$$

(see (3.10)). Hence

$$z_1 = \left(0, t^1_3, -t^1_2, 0,0,0,0,0,0,0,0,0, \frac{1}{2}t^1_2, -\frac{1}{2}t^1_1, 0, -\frac{1}{2}t^1_3, 0, \frac{1}{2}t^1_1\right)^{\mathrm{T}}$$

is a solution of (A.1). In a similar way, we have two other solutions, both of which are linearly independent of each other and independent of z_1:

$$z_2 = \left(0,0,0,0,0,0, t^1_2, -t^1_1, 0, -\frac{1}{2}t^1_3, 0, \frac{1}{2}t^1_1, 0, \frac{1}{2}t^1_3, \frac{1}{2}t^1_2, 0,0,0\right)^{\mathrm{T}},$$

$$z_3 = \left(0,0,0, -t^1_3, 0, t^1_1, 0,0,0, \frac{1}{2}t^1_2, -\frac{1}{2}t^1_1, 0,0,0,0,0, \frac{1}{2}t^1_3, -\frac{1}{2}t^1_2\right)^{\mathrm{T}}.$$

We thus have three linearly independent solutions of (A.1); (I) is established. Any other solution z^* is expressed as a linear combination of z_i's:

$$z^* = -\frac{1}{2}\sum_{s=1}^{3}(\alpha n^2_s - n^1_s)\, z_s,$$

which yields (II).

Case ii ($n^2 = \beta n^1$) : Since $n^2 = \beta n^1$, we can hide n^2 in this case. The $[i,j](k\ell)$-component of a solution is then rewritten as

$$\begin{aligned} T_{[ij](k\ell)} &= \beta t^1_{[i} t^2_{j]} n^1_k n^1_\ell + \frac{1}{4}\Big[\Big\{\delta_{ik}(\beta t^2_j - t^1_j) - \delta_{jk}(\beta t^2_i - t^1_i)\Big\}\, n^1_\ell \\ &\qquad + \Big\{\delta_{i\ell}(\beta t^2_j - t^1_j) - \delta_{j\ell}(\beta t^2_i - t^1_i)\Big\}\, n^1_k\Big]. \end{aligned}$$

It is easy to see that $\tilde{z}_i$ ($i = 1,2,3$) below are three linearly independent solutions of (A.1). Any other solution $\tilde{z}^*$ is expressed as their linear combination:

$$\tilde{z}^* = \frac{1}{2}\sum_{s=1}^{3} n^1_s\, \tilde{z}_s,$$

from which (II) follows.

$$\begin{aligned} \tilde{z}_1^{\mathrm{T}} = \Big(&\beta n^1_1 \check{t}_{23},\ \beta n^1_1 \check{t}_{31} - \Delta t_3,\ \beta n^1_1 \check{t}_{12} + \Delta t_2, \\ &0,0,0, \\ &0,0,0, \\ &0,0,0, \\ &\frac{1}{2}(\beta n^1_3 \check{t}_{23} - \Delta t_2),\ \frac{1}{2}(\beta n^1_3 \check{t}_{31} + \Delta t_1),\ \frac{1}{2}\beta n^1_3 \check{t}_{12}, \\ &\frac{1}{2}(\beta n^1_2 \check{t}_{23} + \Delta t_3),\ \frac{1}{2}\beta n^1_2 \check{t}_{31},\ \frac{1}{2}(\beta n^1_2 \check{t}_{12} - \Delta t_1)\Big), \end{aligned}$$

$$
\begin{aligned}
\tilde{z}_2^{\mathrm{T}} &= (0,0,0, \\
&\quad \beta\, n_2^1\, \breve{t}_{23} + \Delta t_3,\ \beta\, n_2^1\, \breve{t}_{31},\ \beta\, n_2^1\, \breve{t}_{12} - \Delta t_1, \\
&\quad 0,0,0, \\
&\quad \frac{1}{2}(\beta\, n_3^1\, \breve{t}_{23} - \Delta t_2),\ \frac{1}{2}(\beta\, n_3^1\, \breve{t}_{31} + \Delta t_1),\ \frac{1}{2}\beta\, n_3^1\, \breve{t}_{12}, \\
&\quad 0,0,0, \\
&\quad \frac{1}{2}\beta\, n_1^1\, \breve{t}_{23},\ \frac{1}{2}(\beta\, n_1^1\, \breve{t}_{31} - \Delta t_3),\ \frac{1}{2}(\beta\, n_1^1\, \breve{t}_{12} + \Delta t_2)\Big), \\
\tilde{z}_3^{\mathrm{T}} &= (0,0,0, \\
&\quad 0,0,0, \\
&\quad \beta\, n_3^1\, \breve{t}_{23} - \Delta t_2,\ \beta\, n_3^1\, \breve{t}_{31} + \Delta t_1,\ \beta\, n_3^1\, \breve{t}_{12}, \\
&\quad \frac{1}{2}(\beta\, n_2^1\, \breve{t}_{23} + \Delta t_3),\ \frac{1}{2}\beta\, n_2^1\, \breve{t}_{31},\ \frac{1}{2}(\beta\, n_2^1\, \breve{t}_{12} - \Delta t_1), \\
&\quad \frac{1}{2}\beta\, n_1^1\, \breve{t}_{23},\ \frac{1}{2}(\beta\, n_1^1\, \breve{t}_{31} - \Delta t_3),\ \frac{1}{2}(\beta\, n_1^1\, \breve{t}_{12} + \Delta t_2), \\
&\quad 0,0,0)\,,
\end{aligned}
$$

where $\breve{t}_{ij} := t_i^1\, t_j^2 - t_j^1\, t_i^2$ and $\Delta t_i := \beta\, t_i^2 - t_i^1$.

As we can see, when $\boldsymbol{t}^2 = \alpha\, \boldsymbol{t}^1$, $\boldsymbol{n}^2 = \beta\, \boldsymbol{n}^1$ and $\alpha \cdot \beta = 1$ hold, two planes cannot be distinguished. In other words, we are in the situation where the two planes appear to coincide with each other and they completely share the same motion. Therefore, $\dim(\mathrm{Ker}\, A) = 8$ from Conjecture 3.1, which is equivalent to $\mathrm{rank}\, A = 10$. This completes the proof of Theorem 3.3.

B Proof of Theorem 3.4

We first show (1). $P_1 + P_2 \geq 12$ is obvious due to Theorem 3.2.

In order to show the others, we should recall the discussion in Section 2.2 for the case of one plane alone. That is, when one plane is concerned, we can uniquely determine a transformation matrix up to a scale factor if we have "general" four point correspondences. Here "general" implies that no three of the four points in a plane are collinear. This is because three collinear points give only four independent constraint equations in (2.4). Therefore, if we can determine a transformation matrix, then we have at least four point correspondences and we have four points such that no three of them are collinear.

Now we turn to our case where two planes are concerned. Since $T_{(ij)(k\ell)}$ is derived from two transformation matrices, we cannot determine $T_{(ij)(k\ell)}$ unless the two transformation matrices are determined. For each plane, we cannot determine its transformation matrix unless we have "general" four points. Therefore, if $\min(P_1, P_2) < 4$ or we have no "general" four points in each of the two planes, then we cannot determine $T_{(ij)(k\ell)}$. This completes the proof of (1).

We next prove (2). The assumptions indicate that we can uniquely determine M^2 up to a scale factor; whereas we cannot determine M^1. Since each point in plane 1 gives two independent constraint equations in (2.4), we have $(9 - 2P_1)$ linearly independent solutions when we regard (2.4) as a linear homogeneous system (in entries of M^1). Let $\check{M}^1_s$ $(s = 1, 2, \ldots, (9-2P_1))$ be the $(9 - 2P_1)$ linearly independent solutions. Then, any other solution M^1 is expressed as a linear combination of $\check{M}^1_s$'s. Note here that we do not align the entries of M^1 to derive a vector; instead, we directly use each entry and the "solution" is used for each entry in this case. From $\check{M}^1_s$ and M^2, we can derive $\check{T}^s_{(ij)(k\ell)}$ as a counterpart of $T_{(ij)(k\ell)}$. And $\check{T}^s_{(ij)(k\ell)}$ satisfies (3.6). In other words, if we regard (3.6) as a linear homogeneous system in $T_{(ij)(k\ell)}$, then $\check{T}^s_{(ij)(k\ell)}$'s are all of the solutions. Since $\check{T}^s_{(ij)(k\ell)}$ is linear in entries of $\check{M}^1_s$, we can see that $\check{T}^s_{(ij)(k\ell)}$'s are linearly independent solutions and any other solution $T_{(ij)(k\ell)}$ is expressed as the linear combination of $\check{T}^s_{(ij)(k\ell)}$'s. This yields (2).