Internal Use Only
非公開

TR-H-147 0048

Projection Invariants of (*n*-2)-Dimensional Subspaces in *n*-Dimensional Projective Space

Akihiro SUGIMOTO

1995. 5. 11

ATR人間情報通信研究所

〒619-02 京都府相楽郡精華町光台2-2 ☎ 0774-95-1011

ATR Human Information Processing Research Laboratories
2-2, Hikaridai, Seika-cho, Soraku-gun, Kyoto 619-02 Japan
Telephone: +81-774-95-1011
Facsimile: +81-774-95-1008

Projection Invariants of $(n-2)$-Dimensional Subspaces in n-Dimensional Projective Space

Akihiro SUGIMOTO

ATR Human Information Processing Research Laboratories

Soraku-gun, Kyoto 619-02, Japan

e-mail: sugimoto@hip.atr.co.jp

Abstract

When we observe a subject under investigation, we often obtain only a certain part of the original information, that is, information projected, from a space where the original information exists, to its subspace. We are then required to deal with such partial information to investigate the subject. When original information is subject to a given class of admissible transformations, *projection invariants*, functions in terms of the projected information whose values are unaffected by the class of admissible transformations, provide an essential relationship between the original information and the projected one. This paper presents a study on projection invariants under the conditions that the n-dimensional projective space is projected into the $(n-1)$-dimensional space and the class of admissible transformations involves projective transformations. We show the existence of a projection invariant derived from $(n+i+j)$ linear subspaces of dimension $(n-2)$ arranged in the letter H, where i and j are given integers such that $1 \leq i \leq j \leq n-i$. The nonsingularity condition, i.e., the condition under which the projection invariant is nonsingular, is also given.

Key Words: projection invariants, admissible transformations, interpretation vector, intersections of hyperplanes, nonsingularity condition.

Contents

1 Introduction

When we observe a subject under investigation, we often obtain only a certain part of the original information, that is, information projected, from a space where the original information exists, into its subspace. We are then required to deal with such partial information to investigate the subject. For instance, in observing objects in three dimensions, we obtain visual information that was projected onto the retina from the 3-dimensional Euclidean space; we have to recognize the objects by making use of only the projected information on the retina. Since the projection causes a deficiency of information, the problem of recovering the original information is ill-posed; therefore, in general, we cannot uniquely recover the original information from the projected information. In addition, when a transformation acts on the original information, the projected information before and after the transformation differs. In other words, the projected information significantly varies, depending on the transformation to which the original information is subject, even for the same original information. Thus, it is important to find properties, if any, that essentially connect the original information with the projected one.

When original information in a space is subject to a given class of admissible transformations, functions, which are defined in terms of the projected information and whose values are unaffected by the class of admissible transformations, provide an essential relationship between the original information and the projected one. In this paper, we term such functions *projection invariants.* When we cannot directly treat the original information, projection invariants play an important role in investigating properties of the original information. For example, the appearance of an object's shape in the image plane significantly depends on the viewpoint; how to deal with numerous different images of the same object is a crucial problem in computer vision. Since projection invariants, which can be calculated from information in the image plane, have the same value for the same object, they aid in identifying one object out of many and they allow us to effectively tackle the object recognition problem, one of the most important problems in computer vision [2], [3], [7], [10],[11], [12]. (In fact, the importance of projection invariants has been continually emphasized since the origin of the field of computer vision in the 1960s.)

On the other hand, invariants were a very active mathematical subject in the latter half of the 19th century [5]. However, the deficiency of information caused by a projection was not of concern there. Namely, invariants were not derived through projections; they were derived by dealing not with the projected information, but with the original information itself. Therefore, the invariants [1], [4], [8], [9] studied then are nothing but those of admissible transformations themselves. In contrast to this, in practice, we often face situations in which we have to get at the essence of the original information by way of the projected information, and we cannot deal with the original information itself. For example, consider a situation where we have to recognize objects in three dimensions through visual information. Accordingly, investigating the existence of projection invariants is very significant from a practical point of view.

In this paper, we investigate the existence of projection invariants under the conditions that linear subspaces of dimension $(n-2)$ in the $(n-1)$-dimensional projective space were projected from the n-dimensional projective space by the projection of a certain class and the inverse images of these subspaces with respect to the projection are subject to projective general linear transformations (in the n-dimensional projective space). We are mainly interested in deriving projection invariants in a concrete fashion in terms of coefficients of the equations (in the $(n-1)$-dimensional projective space) representing the subspaces of dimension $(n-2)$.

The main theorems, which are given in §3, state that (1) for given integers i and j such that $1 \leq i \leq j \leq n-i$, we have a projection invariant derived from $(n+i+j)$ linear subspaces of dimension $(n-2)$, where the $(n+i+j)$ subspaces are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes arranged in the letter H; and (2) the projection invariant is nonsingular, i.e., well-defined and nondegenerate, iff (COND) below is satisfied by n subspaces among the $(n+i+j)$, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part of H, in the arrangement (we always have four cases).

(COND) Not singular is an $(n+1) \times (n+1)$ matrix whose column vectors are the homogeneous coordinates of $(n+1)$ hyperplanes that determine the n subspaces of dimension $(n-2)$.

In this paper, when an arrangement of hyperplanes or linear subspaces of dimension $(n-2)$ has the same topology as the letter H, we say, "they are arranged in the letter H"; hence, they could n-dimensionally exist. (1) indicates that we have a projection invariant of $(n+i+j)$

subspaces of dimension $(n-2)$ arranged in the letter H (accordingly, the $(n+i+j)$ subspaces could n-dimensionally exist). It should be noted that the number of $(n-2)$-dimensional subspaces in the left part of H is $2i$, whereas that in the right part is $2j$; and, furthermore, the arrangement is symmetrical with respect to the horizontal part of H. In addition, the number of projection invariants of this kind in the n-dimensional projective space is $\lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor\right)$ (see Page 9 for the notation). (2) implies that our projection invariant is almost always nonsingular when we randomly choose $(n+i+j+1)$ hyperplanes in the n-dimensional projective space. This is because the homogeneous coordinates of $(n+1)$ hyperplanes randomly chosen in the n-dimensional projective space are linearly independent in general.

This paper is organized as follows. In §2, we formulate the problem to solve. In §3, the results of this paper, i.e., the existence of projection invariants and the nonsingularity condition for our projection invariants, are presented as two theorems. Their proofs are given in §4. In this paper, we assume that the correspondence of subspaces over projections is known. Henceforth, we sometimes use "invariants" shortly, instead of "projection invariants".

2 Problem Formulation

Let $\mathbf{P}^n$ be the n-dimensional projective space over a certain field $\mathbf{F}$. In applications to computer vision, we usually have $\mathbf{F} = \mathbf{R}$ (the real number field). We discuss the case of $\mathbf{F} = \mathbf{R}$ in this paper, but the same discussion can be applied to other fields. We assume $n \geq 3$ throughout this paper. Note that if not explicitly stated, the coordinates of a point are understood to be homogeneous.

Letting $\boldsymbol{c} = (1, 0, 0, \ldots, 0)^{\mathrm{T}}$ ($\in \mathbf{P}^n$), we consider the set of mappings : $\mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$ as follows.

$$\mathcal{F} \ := \ \{f_P \mid P \in \mathrm{PGL}(n-1)\},$$

where $f_P : \mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$ is represented by an $n \times (n+1)$ matrix F_P:

$$F_P \ = \ \left(\begin{array}{c|c} 0 & \ P \ \end{array} \right) \quad (P \in \mathrm{PGL}(n-1)),$$

and $\mathrm{PGL}(n-1)$ denotes the projective general linear group of degree $(n-1)$ over $\mathbf{R}$. Therefore, when we put $\boldsymbol{x} \in \mathbf{P}^n - \{\boldsymbol{c}\}$ and $\boldsymbol{X} = f_P(\boldsymbol{x})$, then we have

$$\rho \boldsymbol{X} \ = \ F_P \boldsymbol{x} \qquad (\rho \in \mathbf{R}^*),$$

where $\mathbf{R}^*$ denotes the set of nonzero real numbers. In this paper, we are interested in the class $\mathcal{F}$ of mappings : $\mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$; we call an element of $\mathcal{F}$ a *projection*. We assume that we can deal with only $\boldsymbol{X}$, i.e., the image of $\boldsymbol{x}$ projected by f_P, where f_P is derived from a given $P \in \mathrm{PGL}(n-1)$ as seen above. It should be noted that, when we denote by I the unit matrix of degree n, $\forall F_P$ is expressed by

$$F_P \ = \ P F_I.$$

If we restrict $\mathbf{P}^n - \{\boldsymbol{c}\}$ and $\mathbf{P}^{n-1}$ to the n-dimensional vector space over $\mathbf{R}$ that excludes the origin (its coordinates in $\mathbf{P}^n$ are $\boldsymbol{c}$) and hyperplane $x_1 = 0$; and to the $(n-1)$-dimensional vector space over $\mathbf{R}$, respectively, f_I ($\in \mathcal{F}$) then coincides with the central projection where the projection center is the origin and where the projection hyperplane is $x_1 = 1$ (see Fig. 1). Furthermore, in the case of $n = 3$, the central projection realizes the pinhole camera model that is widely used in computer vision.

Let $\mathcal{T}$ be the set of projective transformations that act on an element of $\mathbf{P}^n - \{\boldsymbol{c}\}$:

$$\mathcal{T} \;=\; \{T \mid T \,:\, \mathbf{P}^n - \{\boldsymbol{c}\} \to \mathbf{P}^n,\; T \in \mathrm{PGL}(n)\,\}\,.$$

For $S \subseteq \mathbf{P}^n - \{\boldsymbol{c}\}$, we define

$$\mathcal{T}_S \;:=\; \{T \mid T \in \mathcal{T};\; T(\boldsymbol{x}) \neq \boldsymbol{c},\; \forall \boldsymbol{x} \in S\}.$$

Since $\mathcal{T}_S$ forms a group, we set $\mathcal{T}_S$ to be the class of admissible transformations for S. In addition, we put

$$f_P(S) \;:=\; \bigcup_{\boldsymbol{x} \in S} \{f_P(\boldsymbol{x})\}.$$

In accordance with the notations introduced above, we formulate our problem, namely, the problem of finding a function that is defined in terms of the images of S projected by f_P and whose values are unaffected by $\mathcal{T}_S$, i.e., the class of admissible transformations for S.

Problem 2.1 Let $f_P \in \mathcal{F}$ and S ($\subseteq \mathbf{P}^n - \{\boldsymbol{c}\}$) be given. Find a natural number N and a function $Inv : \overbrace{f_P(S) \times f_P(S) \times \cdots \times f_P(S)}^{N} \longrightarrow \mathbf{R}$ such that,

for $\forall T \in \mathcal{T}_S$,

$$Inv\,(f_P(\boldsymbol{x}),\, f_P(\boldsymbol{x}), \ldots, f_P(\boldsymbol{x})) \;=\; Inv\,(f_P(T(\boldsymbol{x})),\, f_P(T(\boldsymbol{x})), \ldots, f_P(T(\boldsymbol{x})))\,,$$

where $\boldsymbol{x} \in S$. □

Function Inv is a projection invariant under the conditions that the projection is achieved by f_P, and the class of admissible transformations is $\mathcal{T}_S$ for a given S. Our aim in this paper is, for given f_P and S, to find natural number N and function Inv in Problem 2.1.

For $\forall f_P \in \mathcal{F}$, linear subspaces of dimension $(n-2)$ in $\mathbf{P}^n - \{\boldsymbol{c}\}$ are projected into linear subspaces of dimension $(n-2)$ in $\mathbf{P}^{n-1}$ by f_P; and we can deal with the projected subspaces[1]. Hence, we set f_P to be f_{P_*}, that is, f_{P_*} derived from an arbitrary $P_* \in \mathrm{PGL}(n-1)$; and S to be the set whose elements are N linear subspaces, which n-dimensionally exist, of dimension $(n-2)$ in $\mathbf{P}^n - \{\boldsymbol{c}\}$. We then focus on finding a function having the following properties:

[1]Let $f_{P_1}, f_{P_2} \in \mathcal{F}$, then an image of a point in $\mathbf{P}^n - \{\boldsymbol{c}\}$ projected by f_{P_1} is connected to that projected by another projection f_{P_2} through a projective transformation in $\mathbf{P}^{n-1}$ (an element of $\mathrm{PGL}(n-1)$; to be more precise, $P_1 P_2^{-1}$ or $P_2 P_1^{-1}$).

1) it is defined in terms of the coefficients of the equations that determine the N projected subspaces of dimension $(n-2)$; and 2) its value remains invariant even if the inverse images with respect to f_{P_*} are transformed by any admissible transformation, i.e., any element of $\mathcal{T}_S$.

3 Results

The results of this paper are presented as Theorems 3.1 and 3.2. Their proofs are postponed until the next section.

For a linear subspace of dimension $(n-2)$,

$$\sum_{\kappa=0}^{n-1} a_\kappa X_\kappa = 0, \tag{3.1}$$

in $\mathbf{P}^{n-1}$ (its coordinate system is $(X_0, X_1, \ldots, X_{n-1})^{\mathrm{T}}$), where

$$\sum_{\kappa=0}^{n-1} {a_\kappa}^2 \neq 0 \qquad (a_\kappa \in \mathbf{R}),$$

we obtain a vector $(a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$ that is determined by the coefficients of the equation. We call this vector the *interpretation vector* for the subspace. The interpretation vector is the homogeneous coordinates of the subspace.

Remark 3.1 We can only determine vector $(a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$ up to a scaling factor when we actually observe linear subspace (3.1) in $\mathbf{P}^{n-1}$. However, we can eliminate this indeterminacy by setting a criterion such as $a_0 = 1$ or the normalization of the vector. □

A linear subspace of dimension $(n-2)$ in S is uniquely determined as the intersection of a pair of hyperplanes in $\mathbf{P}^n - \{\boldsymbol{c}\}$ (see Fig. 2). Thus, we represent an element of S as a pair of hyperplanes in $\mathbf{P}^n - \{\boldsymbol{c}\}$. For a linear subspace of dimension $(n-2)$ in $\mathbf{P}^n - \{\boldsymbol{c}\}$ determined by two hyperplanes 1 and 2, we denote by $\boldsymbol{n}_{12}$ the interpretation vector for the projected subspace of dimension $(n-2)$ in $\mathbf{P}^{n-1}$.

For two integers i and j such that $1 \le i \le j \le n-i$, we define the following sets of hyperplanes in $\mathbf{P}^n - \{\boldsymbol{c}\}$.

$$\begin{aligned}
\Omega_{\mathrm{L1}} &:= \{\mathrm{L1}_1, \mathrm{L1}_2, \ldots, \mathrm{L1}_i\}, \\
\Omega_{\mathrm{L2}} &:= \{\mathrm{L2}_1, \mathrm{L2}_2, \ldots, \mathrm{L2}_i\}, \\
\Omega_{\mathrm{R1}} &:= \{\mathrm{R1}_j, \mathrm{R1}_{j-1}, \ldots, \mathrm{R1}_1\}, \\
\Omega_{\mathrm{R2}} &:= \{\mathrm{R2}_j, \mathrm{R2}_{j-1}, \ldots, \mathrm{R2}_1\}, \\
\Omega_{\mathrm{C}} &:= \{\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_{n+1-i-j}\},
\end{aligned}$$

where $\mathrm{L1}_\lambda, \mathrm{L2}_\lambda, \mathrm{R1}_\mu, \mathrm{R2}_\mu, \mathrm{C}_\nu$ ($\lambda \in \{1, 2, \ldots, i\}; \mu \in \{1, 2, \ldots, j\}; \nu \in \{1, 2, \ldots, n+1-i-j\}$) are all natural numbers; and any two of Ω_τ ($\tau \in \{\mathrm{L1}, \mathrm{L2}, \mathrm{R1}, \mathrm{R2}, \mathrm{C}\}$) are disjoint. Note that $|\Omega_{\mathrm{L}k}| + |\Omega_{\mathrm{C}}| + |\Omega_{\mathrm{R}\ell}| = n+1$ ($k, \ell \in \{1,2\}$). It is important to remark that we assume that the order of elements of Ω_τ ($\tau \in \{\mathrm{L1}, \mathrm{L2}, \mathrm{R1}, \mathrm{R2}, \mathrm{C}\}$) makes sense. Namely, hyperplanes in Ω_τ are assumed to be aligned with the order of the elements by which Ω_τ is defined. This should be applied to the union of Ω_τ's such as $\Omega_{\mathrm{L1}} \cup \Omega_{\mathrm{C}}$. Here, we assume that $(n+1)$ different hyperplanes $\Omega_{\mathrm{L}k} \cup \Omega_{\mathrm{C}} \cup \Omega_{\mathrm{R}\ell}$ in $\mathbf{P}^n - \{c\}$ are given, where $k, \ell \in \{1,2\}$; and n linear subspaces of dimension $(n-2)$ are observed in $\mathbf{P}^{n-1}$, all of which are the images of the intersections of the adjacent hyperplanes in $\Omega_{\mathrm{L}k} \cup \Omega_{\mathrm{C}} \cup \Omega_{\mathrm{R}\ell}$ projected by f_{P_*}. We then consider the interpretation vectors, $\boldsymbol{n}_{\mathrm{L}k_1\mathrm{L}k_2}, \ldots, \boldsymbol{n}_{\mathrm{L}k_{i-1}\mathrm{L}k_i}, \boldsymbol{n}_{\mathrm{L}k_i\mathrm{C}_1}, \boldsymbol{n}_{\mathrm{C}_1\mathrm{C}_2}, \ldots, \boldsymbol{n}_{\mathrm{C}_{n-i-j}\mathrm{C}_{n+1-i-j}}, \boldsymbol{n}_{\mathrm{C}_{n+1-i-j}\mathrm{R}\ell_j}, \boldsymbol{n}_{\mathrm{R}\ell_j\mathrm{R}\ell_{j-1}}, \ldots, \boldsymbol{n}_{\mathrm{R}\ell_2\mathrm{R}\ell_1}$, for the n intersection subspaces; and define an $n \times n$ matrix $N_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$ whose column vectors are these:

$$
\begin{aligned}
N_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} := \Big[\, & \boldsymbol{n}_{\mathrm{L}k_1\mathrm{L}k_2} \,\Big|\, \cdots \,\Big|\, \boldsymbol{n}_{\mathrm{L}k_{i-1}\mathrm{L}k_i} \,\Big|\, \boldsymbol{n}_{\mathrm{L}k_i\mathrm{C}_1} \,\Big|\, \boldsymbol{n}_{\mathrm{C}_1\mathrm{C}_2} \,\Big|\, \cdots \,\Big| \\
& \boldsymbol{n}_{\mathrm{C}_{n-i-j}\mathrm{C}_{n+1-i-j}} \,\Big|\, \boldsymbol{n}_{\mathrm{C}_{n+1-i-j}\mathrm{R}\ell_j} \,\Big|\, \boldsymbol{n}_{\mathrm{R}\ell_j\mathrm{R}\ell_{j-1}} \,\Big|\, \cdots \,\Big|\, \boldsymbol{n}_{\mathrm{R}\ell_2\mathrm{R}\ell_1} \,\Big].
\end{aligned}
$$

We attach $'$ (prime) to the notations above when an admissible transformation has acted on S.

Theorem 3.1 For two integers i and j such that $1 \le i \le j \le n-i$, let $\Omega_{\mathrm{R1}}, \Omega_{\mathrm{R2}}, \Omega_{\mathrm{C}}, \Omega_{\mathrm{L1}}, \Omega_{\mathrm{L2}}$ above be given sets of hyperplanes in $\mathbf{P}^n - \{c\}$, and let these sets be arranged in the letter H (see Fig. 3). Suppose that $\mathrm{rank} N_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n$ ($k, \ell \in \{1,2\}$). For $(n+i+j)$ linear subspaces of dimension $(n-2)$ that are the intersections of the adjacent hyperplanes in the arrangement, we then have, independent of f_{P_*},

$$
\begin{aligned}
\mathrm{rank} N'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} &= n, \\
\frac{\det N_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}} \cdot \det N_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}}{\det N_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}} \cdot \det N_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}} &= \frac{\det N'_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}} \cdot \det N'_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}}{\det N'_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}} \cdot \det N'_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}}.
\end{aligned}
\tag{3.2}
$$

□

Theorem 3.1 shows that for any element of $\mathcal{F}$ (which is a projection from $\mathbf{P}^n - \{c\}$ to $\mathbf{P}^{n-1}$) there exists a projection invariant, independent of the element,

$$
Inv_{ij} := \frac{\det N_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}} \cdot \det N_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}}{\det N_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}} \cdot \det N_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}} \quad (1 \le i \le j \le n-i)
\tag{3.3}
$$

for $(n+i+j)$ linear subspaces of dimension $(n-2)$, all of which are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes (in $\mathbf{P}^n - \{c\}$) arranged in the letter H (see Fig. 3). It is important to remark that we accordingly have $(n+i+j)$ linear subspaces of dimension $(n-2)$ arranged in the letter H (hence, the $(n+i+j)$ subspaces could n-dimensionally exist); and also to remark that the number of subspaces in the left-upper part of H is equal to that in the left-lower part: i. Whereas, the number of subspaces in the right-upper part of H is equal to that in the right-lower part: j. Namely, the arrangement is symmetrical with respect to the horizontal part of H.

In summary, for $\forall f_P \in \mathcal{F}$, when we set S to be the set whose elements are N linear subspaces of dimension $(n-2)$ in $\mathbf{P}^n - \{\boldsymbol{c}\}$ arranged in the letter H, N and Inv in Problem 2.1 are respectively given by $N = n+i+j$ and (3.3), where i and j are given integers such that $1 \leq i \leq j \leq n-i$. We should note that $n+2 \leq N \leq 2n$.

Remark 3.2 Since $i+j = n$ is possible, we could have $|\Omega_{\mathrm{C}}| = 1$. Namely, for the linear subspaces of dimension $(n-2)$ arranged in the letter H, the part that corresponds to the horizontal part of H could be empty. □

For each i, we have $\xi_i = (n-2i+1)$ invariants. Taking symmetry into consideration, i can be any of $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Hence, the number[2] Ξ of invariants of this kind in $\mathbf{P}^n - \{\boldsymbol{c}\}$ is given by

$$\begin{aligned} \Xi &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \xi_i \\ &= \lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor \right), \end{aligned}$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the maximum integer not greater than $\frac{n}{2}$.

Furthermore, we give the nonsingularity condition for Inv_{ij} $(1 \leq i \leq j \leq n-i)$, i.e., the necessary and sufficient condition making Inv_{ij} nonsingular. Here, we define "an invariant is nonsingular" as "the value of the invariant is not 0, ∞ or $0/0$". Nonsingularity can be regarded as nondegeneracy and well-definedness. As we can see, the nonsingularity condition for an invariant ensures that the values of the invariant are numerically stable when they are calculated in practical situations. The next theorem indicates that the nonsingularity condition for invariant Inv_{ij} is almost always satisfied, when we randomly choose $(n+i+j+1)$ hyperplanes

[2]In particular, we have $\lfloor \frac{n}{2} \rfloor$ invariants for $2n$ linear subspaces of dimension $(n-2)$; whereas we have only one invariant for $(n+2)$ linear subspaces of dimension $(n-2)$.

in $\mathbf{P}^n - \{c\}$. This is because the homogeneous coordinates of $(n+1)$ hyperplanes that were randomly chosen in $\mathbf{P}^n - \{c\}$, are linearly independent in general. Note that $(n+i+j+1)$ hyperplanes arranged in the letter H could n-dimensionally exist.

Theorem 3.2 [Nonsingularity condition]

Let $(n+i+j+1)$ hyperplanes, where $(n+i+j)$ linear subspaces of dimension $(n-2)$ exist, be arranged in the letter H (see Fig. 3). Inv_{ij} in (3.3) is nonsingular iff (COND) below is satisfied by n subspaces among the $(n+i+j)$ subspaces, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part Ω_{C} of H, in the arrangement (we always have four cases).

(COND) Not singular is an $(n+1) \times (n+1)$ matrix whose column vectors are the homogeneous coordinates (in $\mathbf{P}^n - \{c\}$) of $(n+1)$ hyperplanes that determine the n subspaces of dimension $(n-2)$. □

4 Proofs

The proofs for Theorems 3.1 and 3.2 are given here.

For a linear subspace of dimension $(n-2)$ in $\mathbf{P}^{n-1}$, we consider the hyperplane on which both $\boldsymbol{c}$ and the subspace exist (see Fig. 4). We refer to this hyperplane as the *interpretation hyperplane* for the subspace. It is easy to see that for a linear subspace of dimension $(n-2)$ in $\mathbf{P}^{n-1}$, any linear subspace of dimension $(n-2)$ in $\mathbf{P}^{n}-\{\boldsymbol{c}\}$ that exists in its interpretation hyperplane is projected to the subspace (in $\mathbf{P}^{n-1}$). It should be remarked that we use the interpretation hyperplane *of* (instead of "for") a linear subspace of dimension $(n-2)$ when the subspace is not in $\mathbf{P}^{n-1}$, but in $\mathbf{P}^{n}-\{\boldsymbol{c}\}$.

For a linear subspace of dimension $(n-2)$ in $(\mathbf{P}^{n-1})$, its interpretation vector is obtained as a result of applying[3] $f_{P_*^{-\mathrm{T}}}$ to the homogeneous coordinates of the interpretation hyperplane for the subspace. This can be understood in the following way. Namely, for a linear subspace (3.1) of dimension $(n-2)$, let $\boldsymbol{X}$ $(\boldsymbol{X}\neq\boldsymbol{0})$ be the coordinates (in $\mathbf{P}^{n-1}$) of any point in the subspace and put $\tilde{\boldsymbol{X}} = P_*^{-1}\boldsymbol{X}\left(=(\tilde{X}_0,\tilde{X}_1,\ldots,\tilde{X}_{n-1})^{\mathrm{T}}\right)$. Then, $(1,\tilde{X}_0,\tilde{X}_1,\ldots,\tilde{X}_{n-1})^{\mathrm{T}}$ is the inverse image of $\boldsymbol{X}$ with respect to f_{P_*}. (In other words, a point in $\mathbf{P}^{n}-\{\boldsymbol{c}\}$ with coordinates $(1,\tilde{X}_0,\tilde{X}_1,\ldots,\tilde{X}_{n-1})^{\mathrm{T}}$ is projected to the point (with coordinates $\boldsymbol{X}$) in $\mathbf{P}^{n-1}$ by f_{P_*}.) Moreover, put $\boldsymbol{a}=(a_0,a_1,\ldots,a_{n-1})^{\mathrm{T}}$ and $\tilde{\boldsymbol{a}}=P_*^{\mathrm{T}}\boldsymbol{a}\left(=(\tilde{a}_0,\tilde{a}_1\ldots,\tilde{a}_{n-1})^{\mathrm{T}}\right)$, then (3.1) is rewritten as

$$(0,\tilde{a}_0,\tilde{a}_1,\ldots,\tilde{a}_{n-1})^{\mathrm{T}}\cdot(1,\tilde{X}_0,\tilde{X}_1,\ldots,\tilde{X}_{n-1})^{\mathrm{T}} \;=\; 0. \tag{4.1}$$

(4.1) represents the hyperplane on which both $\boldsymbol{c}$ and subspace (3.1) exist. Hence, (4.1) is the interpretation hyperplane for subspace (3.1); $(0,\tilde{a}_0,\tilde{a}_1,\ldots,\tilde{a}_{n-1})^{\mathrm{T}}$ is the homogeneous coordinates (or equivalently the normal vector) of the interpretation hyperplane. From $\boldsymbol{a}=P_*^{-\mathrm{T}}\tilde{\boldsymbol{a}}=P_*^{-\mathrm{T}}F_I\,(0,\tilde{a}_0,\tilde{a}_1,\ldots,\tilde{a}_{n-1})^{\mathrm{T}}$, we can see that interpretation vector $\boldsymbol{a}$ is obtained by applying $f_{P_*^{-\mathrm{T}}}$ to the homogeneous coordinates of the interpretation hyperplane for subspace (3.1).

As seen above, we have represented a linear subspace of dimension $(n-2)$ in S as a pair of hyperplanes in $\mathbf{P}^{n}-\{\boldsymbol{c}\}$. Thus, for an intersection subspace of two hyperplanes, we next consider the relationship between the interpretation vector for the projected intersection subspace and the homogeneous coordinates (in $\mathbf{P}^{n}-\{\boldsymbol{c}\}$) of the two hyperplanes. Let hyperplane

[3] For a square matrix P, $P^{-\mathrm{T}}$ is $(P^{\mathrm{T}})^{-1}$ or equivalently $(P^{-1})^{\mathrm{T}}$.

η ($\eta = 1, 2$) in $\mathbf{P}^n - \{\boldsymbol{c}\}$ be the set of points with coordinates $\boldsymbol{x}$ satisfying

$$\boldsymbol{a}_\eta \cdot \boldsymbol{x} \;=\; 0,$$

where

$$\boldsymbol{a}_\eta = (a_{\eta_0}, a_{\eta_1}, \ldots, a_{\eta_n})^{\mathrm{T}}; \quad a_{\eta_0} \in \mathbf{R}^*, \quad a_{\eta_\kappa} \in \mathbf{R}\ (\kappa \in \{1, 2, \ldots, n\}).$$

Then $\boldsymbol{x}$, the coordinates of a point that exists on both hyperplanes 1 and 2 (hence, the point exists in the $(n-2)$-dimensional intersection subspace of the two hyperplanes), satisfies

$$\sum_{\eta=1}^{2} \mu_\eta \left(\boldsymbol{a}_\eta \cdot \boldsymbol{x}\right) \;=\; 0, \tag{4.2}$$

where μ_η ($\eta = 1, 2$) are real numbers. By fixing the values of μ_η ($\eta = 1, 2$) so that $\boldsymbol{c}$ satisfies (4.2), we obtain the interpretation hyperplane of the intersection subspace of two hyperplanes 1 and 2:

$$\left(a_{2_0} \boldsymbol{a}_1 - a_{1_0} \boldsymbol{a}_2\right) \cdot \boldsymbol{x} \;=\; 0.$$

Therefore, $a_{2_0}\boldsymbol{a}_1 - a_{1_0}\boldsymbol{a}_2$ is the homogeneous coordinates of the interpretation hyperplane of the intersection subspace of hyperplanes 1 and 2; we obtain $F_{P_*^{-\mathrm{T}}}\left(a_{2_0}\boldsymbol{a}_1 - a_{1_0}\boldsymbol{a}_2\right)$ when we observe the subspace in $\mathbf{P}^n - \{\boldsymbol{c}\}$ determined by $\boldsymbol{a}_\eta$'s ($\eta = 1, 2$). It is important to note that we have indeterminacy of a scaling factor between the vector $F_{P_*^{-\mathrm{T}}}\left(a_{2_0}\boldsymbol{a}_1 - a_{1_0}\boldsymbol{a}_2\right)$ and the interpretation vector $\boldsymbol{n}_{12}$ that we actually obtain as a result of observing the subspace. Therefore, defining

$$\boldsymbol{a}_{12} \;:=\; F_{P_*^{-\mathrm{T}}}\left(a_{2_0}\boldsymbol{a}_1 - a_{1_0}\boldsymbol{a}_2\right),$$

we have

$$\boldsymbol{n}_{12} \;=\; \rho_{(1,2)}\, \boldsymbol{a}_{12} \quad (\rho_{(1,2)} \in \mathbf{R}^*). \tag{4.3}$$

Here, $\rho_{(1,2)}$ is a scaling factor and its value is not known. In line with treating $\boldsymbol{n}_{12}$, we define an $n \times n$ matrix $M_{\Omega_{\mathrm{L}k}, \Omega_{\mathrm{C}}, \Omega_{\mathrm{R}\ell}}$ ($k, \ell \in \{1, 2\}$) as a counterpart of $N_{\Omega_{\mathrm{L}k}, \Omega_{\mathrm{C}}, \Omega_{\mathrm{R}\ell}}$:

$$\begin{aligned} M_{\Omega_{\mathrm{L}k}, \Omega_{\mathrm{C}}, \Omega_{\mathrm{R}\ell}} \;:=\; \Big[\, & \boldsymbol{a}_{\mathrm{L}k_1\mathrm{L}k_2} \,\Big|\, \cdots \,\Big|\, \boldsymbol{a}_{\mathrm{L}k_{i-1}\mathrm{L}k_i} \,\Big|\, \boldsymbol{a}_{\mathrm{L}k_i\mathrm{C}_1} \,\Big|\, \boldsymbol{a}_{\mathrm{C}_1\mathrm{C}_2} \,\Big|\, \cdots \,\Big| \\ & \boldsymbol{a}_{\mathrm{C}_{n-i-j}\mathrm{C}_{n+1-i-j}} \,\Big|\, \boldsymbol{a}_{\mathrm{C}_{n+1-i-j}\mathrm{R}\ell_j} \,\Big|\, \boldsymbol{a}_{\mathrm{R}\ell_j\mathrm{R}\ell_{j-1}} \,\Big|\, \cdots \,\Big|\, \boldsymbol{a}_{\mathrm{R}\ell_2\mathrm{R}\ell_1} \,\Big]. \end{aligned} \tag{4.4}$$

(4.3) and (4.4) yield

$$\det N_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} \;=\; P_{k\ell}\cdot \det M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}, \tag{4.5}$$

where

$$P_{k\ell} \;:=\; \rho_{(\mathrm{L}k_i,\mathrm{C}_1)}\cdot \rho_{(\mathrm{C}_{n+1-i-j},\mathrm{R}\ell_j)}\cdot \prod_{\kappa=1}^{i-1}\rho_{(\mathrm{L}k_\kappa,\mathrm{L}k_{\kappa+1})}\cdot \prod_{\kappa=1}^{n-i-j}\rho_{(\mathrm{C}_\kappa,\mathrm{C}_{\kappa+1})}\cdot \prod_{\kappa=1}^{j-1}\rho_{(\mathrm{R}\ell_{\kappa+1},\mathrm{R}\ell_\kappa)}.$$

We again attach $'$ (prime) to the notations above when an admissible transformation has acted on S. Accordingly, we obtain[4]

$$\text{LHS of (3.2)} \;=\; \frac{\det M_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}\cdot \det M_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}}{\det M_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}\cdot \det M_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}},$$

$$\text{RHS of (3.2)} \;=\; \frac{\det M'_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}\cdot \det M'_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}}{\det M'_{\Omega_{\mathrm{L1}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R2}}}\cdot \det M'_{\Omega_{\mathrm{L2}},\Omega_{\mathrm{C}},\Omega_{\mathrm{R1}}}}.$$

Now, to prove Theorem 3.1 it suffices to show the following lemma (applying the results of four combinations of k and ℓ in Lemma 4.1 to the two equations above, completes the proof of Theorem 3.1).

Lemma 4.1 Let a point (with coordinates $\boldsymbol{x}$) in S change its coordinates to $\boldsymbol{x}'$ after an admissible transformation T $(\in \mathcal{T}_S)$ as follows:

$$\lambda\,\boldsymbol{x}' \;=\; T\,\boldsymbol{x} \qquad (\lambda \in \mathbf{R}^*). \tag{4.6}$$

For $k,\ell \in \{1,2\}$ we have, independent of f_{P_*},

$$\operatorname{rank} M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n \;\Longrightarrow\; \operatorname{rank} M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n, \tag{4.7}$$

$$\det T\cdot \prod_{\kappa\in\Gamma_{k\ell}} a_{\kappa 0}\cdot \det M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} \;=\; \prod_{\eta\in\Upsilon_{k\ell}} v_\eta\cdot \prod_{\kappa\in\Gamma_{k\ell}} a'_{\kappa 0}\cdot \det M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}, \tag{4.8}$$

where

$$\Upsilon_{k\ell} \;:=\; \Omega_{\mathrm{L}k}\cup\Omega_{\mathrm{C}}\cup\Omega_{\mathrm{R}\ell},$$

$$\Gamma_{k\ell} \;:=\; \Omega_{\mathrm{L}k}\cup\Omega_{\mathrm{C}}\cup\Omega_{\mathrm{R}\ell} - \{\mathrm{L}k_1,\mathrm{R}\ell_1\}.$$

[4] LHS and RHS stand for the left-hand side and the right-hand side, respectively.

Proof: It follows from the definition of $M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$ that

$$(4.9) \qquad \det M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = \frac{1}{\det P_*} \cdot (-1)^{\mathrm{Mod}_2(n)} \cdot \prod_{\kappa \in \Gamma_{k\ell}} a_{\kappa 0} \cdot \det A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}.$$

Here, $(n+1)\times(n+1)$ matrix $A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$ is defined by

$$A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} := [\, \boldsymbol{a}_{\mathrm{L}k_1} \mid \boldsymbol{a}_{\mathrm{L}k_2} \mid \cdots \mid \boldsymbol{a}_{\mathrm{L}k_i} \mid \boldsymbol{a}_{\mathrm{C}_1} \mid \cdots \mid \boldsymbol{a}_{\mathrm{C}_{n+1-i-j}} \mid \boldsymbol{a}_{\mathrm{R}\ell_j} \mid \cdots \mid \boldsymbol{a}_{\mathrm{R}\ell_1} \,];$$

and, for a natural number n, Mod_2 is a function such that

$$\mathrm{Mod}_2(n) = \begin{cases} 0 & n\text{: even} \\ 1 & n\text{: odd.} \end{cases}$$

Similarly, after admissible transformation T, we define $(n+1)\times(n+1)$ matrix

$$A'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} := [\, \boldsymbol{a}'_{\mathrm{L}k_1} \mid \boldsymbol{a}'_{\mathrm{L}k_2} \mid \cdots \mid \boldsymbol{a}'_{\mathrm{L}k_i} \mid \boldsymbol{a}'_{\mathrm{C}_1} \mid \cdots \mid \boldsymbol{a}'_{\mathrm{C}_{n+1-i-j}} \mid \boldsymbol{a}'_{\mathrm{R}\ell_j} \mid \cdots \mid \boldsymbol{a}'_{\mathrm{R}\ell_1} \,].$$

When a point in S is subject to a transformation of (4.6), $\boldsymbol{a}_\eta$, the coordinates of hyperplane η, is subject to

$$(4.10) \qquad \boldsymbol{a}'_\eta = v_\eta T^{-\mathrm{T}} \boldsymbol{a}_\eta,$$

where v_η ($\eta \in \Upsilon_{k\ell}$) are nonzero real numbers. We then obtain

$$(4.11) \qquad \det M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = \frac{(-1)^{\mathrm{Mod}_2(n)}}{\det P_*} \cdot \prod_{\kappa \in \Gamma_{k\ell}} a'_{\kappa 0} \cdot \det A'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$$
$$= \frac{(-1)^{\mathrm{Mod}_2(n)}}{\det P_*} \cdot \frac{1}{\det T} \cdot \prod_{\eta \in \Upsilon_{k\ell}} v_\eta \cdot \prod_{\kappa \in \Gamma_{k\ell}} a'_{\kappa 0} \cdot \det A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}},$$

since (4.10) yields

$$\det A'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = \frac{1}{\det T} \cdot \prod_{\eta \in \Upsilon_{k\ell}} v_\eta \cdot \det A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}.$$

Note that $a'_{\kappa 0} \neq 0$ ($\kappa \in \Gamma_{k\ell}$) is satisfied since $T \in \mathcal{T}_S$. (4.9) and (4.11) immediately yield (4.8). It is clear that (4.8) is independent of f_{P_*}.

Since $a_{\kappa 0} \neq 0$ ($\kappa \in \Gamma_{k\ell}$), it follows from (4.9) that $\mathrm{rank} M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n$ is equivalent to $\mathrm{rank} A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n+1$. Hence, we have (4.7) from (4.11) since $v_\eta \neq 0$ ($\eta \in \Upsilon_{k\ell}$) and $a'_{\kappa 0} \neq 0$ ($\kappa \in \Gamma_{k\ell}$). □

Remark 4.1 (4.9) shows that $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ is equivalent to $\det M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ $(k, \ell \in \{1,2\})$. Namely, n subspaces in $\mathbf{P}^n - \{\boldsymbol{c}\}$ (the intersection subspaces of the adjacent hyperplanes in $\Omega_{Lk} \cup \Omega_C \cup \Omega_{R\ell}$) share a common point in $\mathbf{P}^{n-1}$ through the projection f_{P_*} iff $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ (see Observation 4.1 below). We assume that $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} \neq 0$ is satisfied by $(n+1)$ hyperplanes that determine these n subspaces (intuitively, this assumption is equivalent to the random choice of $(n+1)$ hyperplanes). Moreover, Lemma 4.1 indicates that if $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} \neq 0$ holds, we can guarantee that these n subspaces after any admissible transformations never share a common point in $\mathbf{P}^{n-1}$ through the projection f_{P_*}. □

We now turn to the proof of Theorem 3.2. From (3.3) it is easy to see that Inv_{ij} is nonsingular iff the values of the determinants of $N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ $(k, \ell \in \{1,2\})$ are not zero. Hence, the necessary and sufficient condition under which Inv_{ij} is nonsingular is that the values of the determinants of $M_{\Omega_{Rk},\Omega_C,\Omega_{R\ell}}$ are not zero (see (4.5)). Observation 4.1 below indicates that when n linear subspaces of dimension $(n-2)$ in $\mathbf{P}^{n-1}$ do not share a common point, the value of the determinant of $M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ is never zero. This argument yields Theorem 3.2 (see Remark 4.1).

Observation 4.1 Let n different linear subspaces τ $(\tau = 1, 2, \ldots, n)$ of dimension $(n-2)$ in $\mathbf{P}^{n-1}$ be

$$\sum_{\kappa=0}^{n-1} a_{\tau_\kappa} X_\kappa = 0,$$

where

$$\sum_{\kappa=0}^{n-1} {a_{\tau_\kappa}}^2 \neq 0 \qquad (a_{\tau_\kappa} \in \mathbf{R}).$$

They do not share a common point in $\mathbf{P}^{n-1}$ iff

$$\det \begin{bmatrix} a_{1_0} & \cdots & a_{\tau_0} & \cdots & a_{n_0} \\ a_{1_1} & \cdots & a_{\tau_1} & \cdots & a_{n_1} \\ \vdots & & \vdots & & \vdots \\ a_{1_\kappa} & \cdots & a_{\tau_\kappa} & \cdots & a_{n_\kappa} \\ \vdots & & \vdots & & \vdots \\ a_{1_{n-1}} & \cdots & a_{\tau_{n-1}} & \cdots & a_{n_{n-1}} \end{bmatrix} \neq 0.$$

□

5 Conclusion

We have investigated the existence of projection invariants under the conditions that the projection from $\mathbf{P}^n - \{c\}$ to $\mathbf{P}^{n-1}$ is achieved by an element of $\mathcal{F}$, and the class of admissible transformations is $\mathcal{T}_S$, where S is the set whose elements are linear subspaces of dimension $(n-2)$ in $\mathbf{P}^n - \{c\}$. For given integers i and j such that $1 \leq i \leq j \leq n-i$, we derived projection invariant, independent of the element of $\mathcal{F}$, Inv_{ij} in (3.3) from $(n+i+j)$ subspaces of dimension $(n-2)$, where these subspaces are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes arranged in the letter H. Accordingly, the $(n+i+j)$ subspaces are also arranged in the letter H (hence, the $(n+i+j)$ subspaces could n-dimensionally exist). Note that the number of subspaces in the left-upper part of H is i, whereas that in the right-upper part is j; and the arrangement is symmetrical with respect to the horizontal part. Let us remark again that the horizontal part could be empty since $i+j=n$ is possible. In addition, the number of invariants of this kind in $\mathbf{P}^n - \{c\}$ is $\lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor\right)$.

Furthermore, the nonsingularity condition for Inv_{ij}, i.e., the necessary and sufficient condition making Inv_{ij} nonsingular, was given. Inv_{ij} is nonsingular iff (COND) below is satisfied by n subspaces among the $(n+i+j)$ subspaces, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part Ω_{C} of H, in the arrangement H above (we always have four cases).

(COND) Not singular is an $(n+1) \times (n+1)$ matrix whose column vectors are the homogeneous coordinates (in $\mathbf{P}^n - \{c\}$) of $(n+1)$ hyperplanes that determine the n subspaces of $(n-2)$ dimensions.

The nonsingularity condition guarantees that Inv_{ij} is not only well-defined but nondegenerate; it also ensures that the values of Inv_{ij} are numerically stable when they are calculated in practical situations. We should remark that this condition is almost always satisfied when we randomly choose $(n+i+j+1)$ hyperplanes in $\mathbf{P}^n - \{c\}$.

The values of Inv_{ij} depend on the order of linear subspaces of dimension $(n-2)$ in the computation (see (3.3)). Namely, a different ordering of the subspaces, i.e., associating indices with the subspaces in a different way, can yield different values of Inv_{ij}. If the values of Inv_{ij} are insensitive to the order, then we need not establish the subspace correspondence in a certain sense. We can thus avoid being concerned with every possible ordering of the subspaces from which Inv_{ij} is computed; projection invariants get more useful in applications. To make them insensitive to the order, we should derive order-independent projection invariants from Inv_{ij} such as j-invariant [8] in the case of four collinear points or p^2-invariant [6] in the case of five coplanar points. Since Inv_{ij} is in a similar form of cross ratio, it would be possible to derive an order-independent invariant from Inv_{ij}. Elaboration of deriving such invariant is left open for future research. Also left for future research is investigating the existence of projection invariants under another projection class.

Acknowledgments

The author is especially thankful to Kazuo Murota of Research Institute for Mathematical Science of Kyoto University for discussion as well as comments on this work. He extends his thanks to Yoh'ichi Tohkura and Shigeru Akamatsu of ATR Human Information Processing Research Laboratories for their encouragement.

References

[1] H. S. M. Coxeter: *Projective Geometry* (2nd edition), Springer-Verlag, New York, U.S.A., 1987.

[2] R. O. Duda and P. E. Hart: *Pattern Classification and Scene Analysis*, Wiley, New York, U.S.A., 1973.

[3] O. Faugeras: *Three-Dimensional Computer Vision: A Geometric Viewpoint*, MIT Press, Cambridge, Massachusetts, U.S.A., 1993.

[4] J. Harris: *Algebraic Geometry*, Graduate Texts in Mathematics 133, Springer-Verlag, New York, U.S.A., 1992.

[5] F. Klein: Vergleichende Betrachtungen über neuere geometrische Forschungen, *Math. Ann.*, 43, 63–100 (1893).

[6] P. Meer, S. Ramakrishna and R. Lenz: Correspondence of Coplanar Features through P^2-Invariant Representations, *Proc. of the 12th Int. Conf. on Pattern Recognition*, Vol. 1, 196–200, 1994.

[7] J. L. Mundy and A. Zisserman eds.: *Geometric Invariance in Computer Vision*, MIT Press, Cambridge, Massachusetts, U.S.A., 1992.

[8] A. N. Parshin and I. R. Shafarevich eds.: *Algebraic Geometry IV*, Encyclopedia of Mathematical Sciences Vol. 55, Springer-Verlag, Berlin, Germany, 1994.

[9] J. G. Semple and G. T. Kneebone: *Algebraic Projective Geometry*, Clarendon Press, Oxford, U.K., 1952 (reprinted in 1979).

[10] A. Sugimoto: Geometric Invariant of Noncoplanar Lines in a Single View, *Proc. of the 12th Int. Conf. on Pattern Recognition*, Vol. 1, 190–195, 1994.

[11] A. Sugimoto: Invariants within the Context of Vision and Their Applications (in Japanese), *SIG-CV-93, Information Processing Society of Japan*, 19–34 (1995).

[12] I. Weiss: Geometric Invariants and Object Recognition, *Int. J. of Computer Vision*, 10, 3, 207–231 (1993).

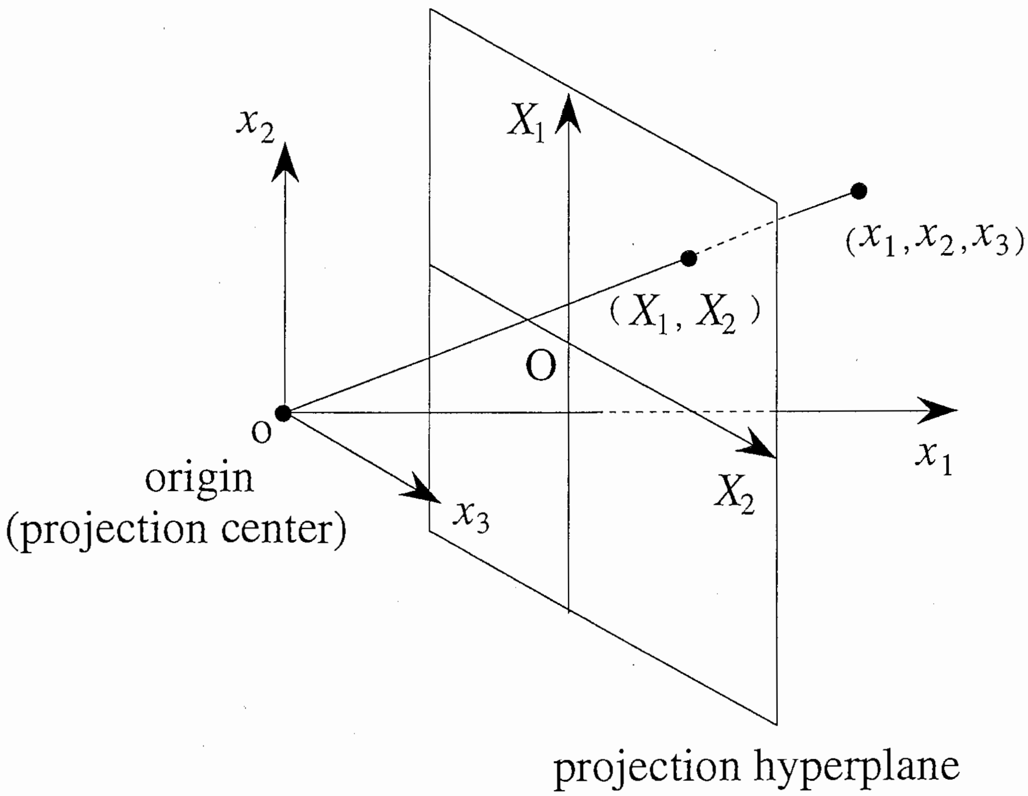

Fig. 1: Central projection attached at origin o ($n = 3$)

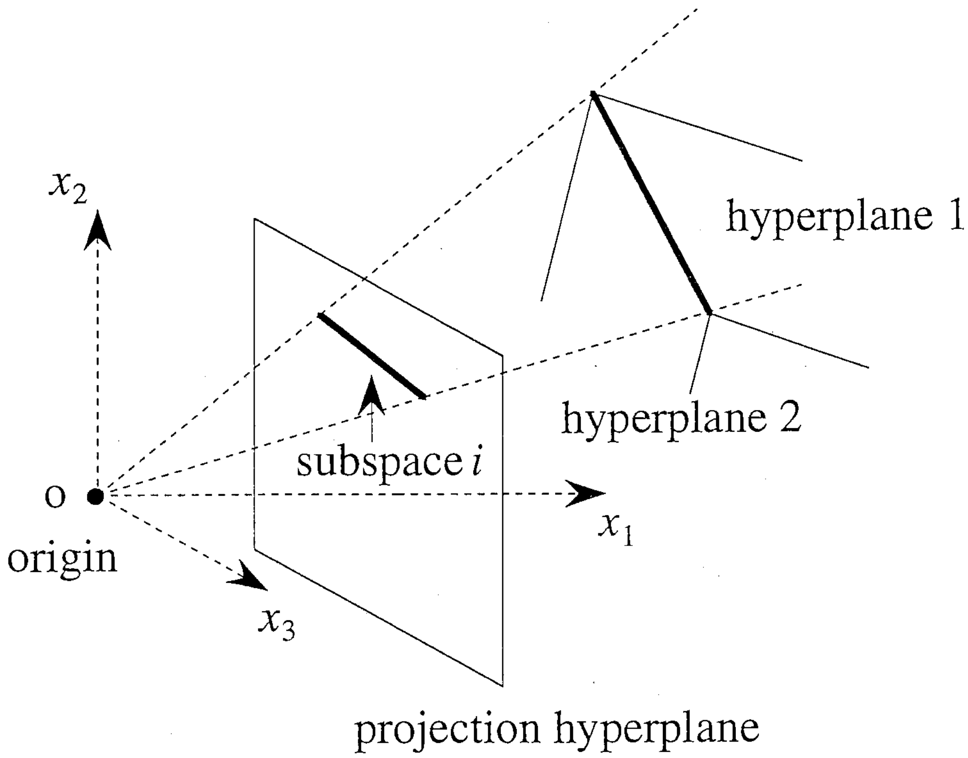

Fig. 2: Linear subspace of dimension $(n-2)$ determined as a pair of hyperplanes $(n=3)$

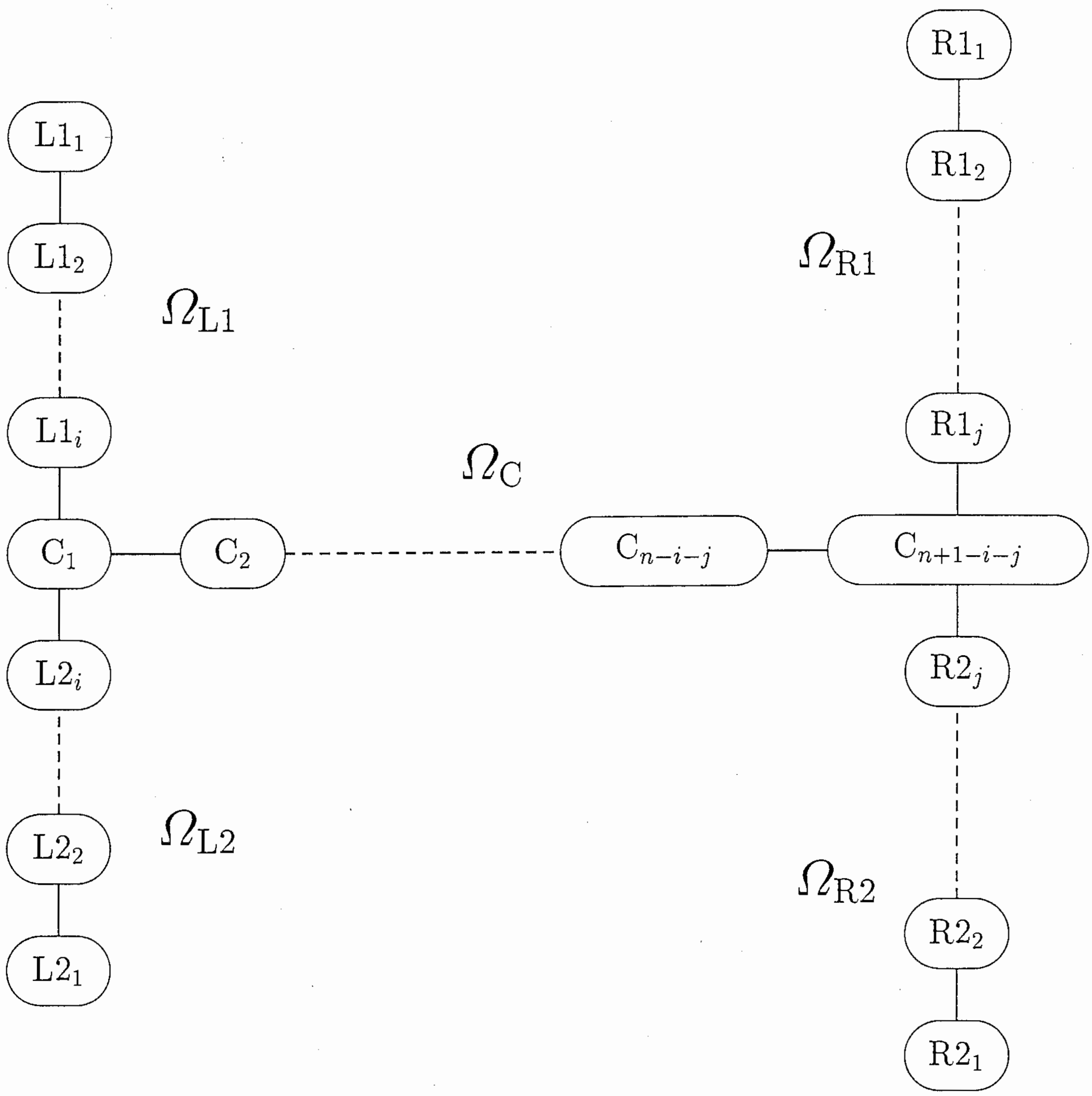

Fig. 3: Arrangement H of $(n+i+j+1)$ hyperplanes and linear subspaces of dimension $(n-2)$ as the intersections of the adjacent hyperplanes (the numbers in ellipses represent hyperplanes; the lines and the dashed lines represent linear subspaces of dimension $(n-2)$)

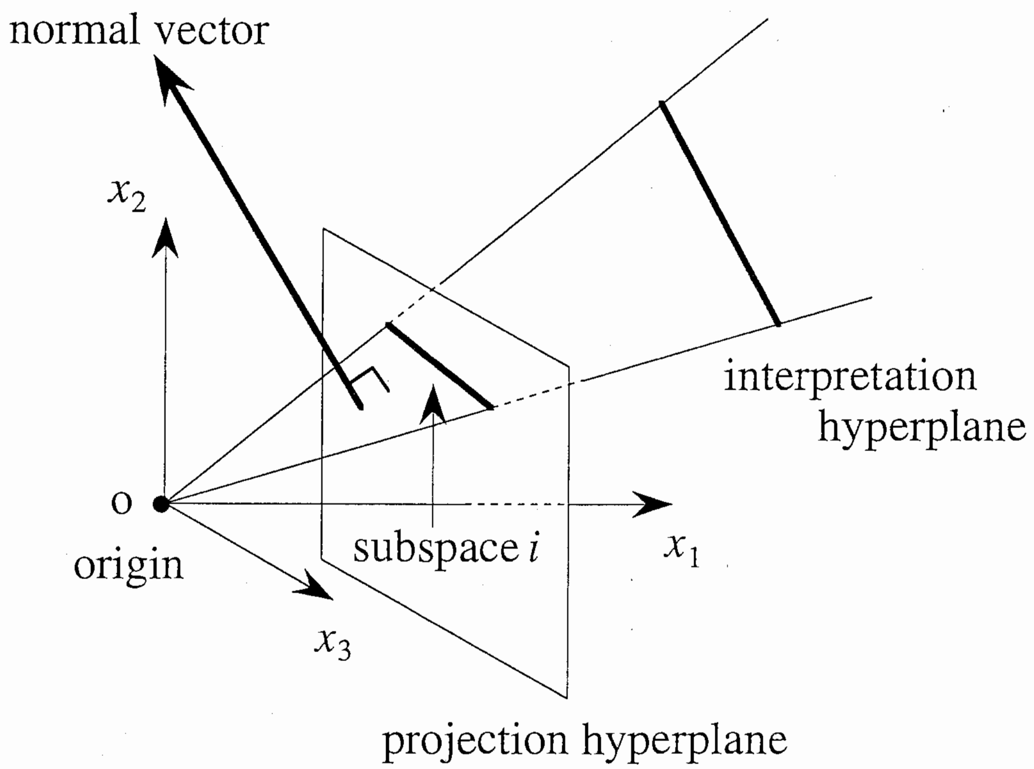

Fig. 4: Subspace i and the homogeneous coordinates (or equivalently the normal vector) of its interpretation hyperplane ($n = 3$)