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# Projective Invariants of Noncoplanar Lines Derived from a Single View

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# Projective Invariants of Noncoplanar Lines Derived from a Single View

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**Abstract**

The importance of projective invariants to many machine vision tasks, such as model-based recognition, has been emphasized because an object generically has its own value for an invariant. A number of recent studies on projective invariants in a single view concentrate on coplanar objects: coplanar points, coplanar lines, coplanar points and lines, coplanar conics, etc. Therefore, it is essentially only to 2-D objects that we can apply methods using invariants. This paper presents a study on projective invariants of noncoplanar objects, that is, 3-D objects. Two new projective invariants are derived from noncoplanar lines in a single view: one from five lines on two planes and the other from six lines on three planes. The conditions under which they are nonsingular are also described. In addition, we present some experimental results with real images and we find that the values of the invariants over a number of viewpoints remain stable even for noisy images. Hence, we no longer need assume coplanar objects; we can directly treat 3-D objects to calculate invariants.

**Key Words:** projective invariants, noncoplanar lines, nonsingularity, 3-D object recognition.

# Contents

## 1 Introduction

An invariant of a geometric configuration is a function of the configuration whose value is unaffected by a change in viewpoint and that is defined in terms of coordinates of image points, or coefficients of equations that represent image lines (or curves). In addition, its value generically depends on an object in three dimensions. Though the appearance of an object's shape significantly depends on the viewpoint, by taking advantage of invariants, we can entirely avoid the crucial problem of how to deal with numerous different images of the same object. Accordingly, we can effectively tackle a number of machine vision tasks, such as object recognition or shape description [7], [18], [29], [30]. For instance, a classical approach to model-based object recognition [2] is divided into two main procedures: for a given image, (i) determine the position of an object relative to a viewpoint, i.e., *pose determination*; and then (ii) compare the given image of an object with every image stored in a library of models to identify the object. In this approach, we obviously face a computational complexity problem in identifying the object even for a small library of models. However, if we use invariants, then attaching invariant values to images in the library makes it possible to directly compare the given image with one in the library without executing procedure (i), and, furthermore, allows a reduction in the number of images to be compared [9], [25]. As for the problem of model description, how to describe an object's shape is the main concern. Using invariant shape descriptors is definitely more efficient since such descriptions are unaffected by a change in viewpoint. As has been seen, invariants are not only important, but are also readily applicable to problems in the field of computer vision.

From this point of view, the importance of invariants has been continually emphasized since the origin of the field of computer vision in the 1960s. Going back further, invariants were a very active mathematical subject in the latter half of the 19th century [15]. However, the deficiency of information caused by a projection was not of concern there. Namely, invariants [11], [19] were not derived through projections; they were derived on the assumption that 3-D information can be directly treated. Since we can actually treat only projected 2-D information, until recently just one invariant [5], the cross ratio of four collinear points, was used in computer vision. Only over the past few years have we highlighted other invariants.

During this time, several invariants were derived and are now being used in machine vision applications. They include, for instance, two invariants of five coplanar points [1], two invariants of five coplanar lines [18], one invariant of two points coplanar with two lines [31], two invariants of two coplanar conics [9], [10], [20] and one invariant of two points coplanar with a conic [18]. If we assume points, lines or conics in 3-D to be coplanar, there are such invariants as listed above. This is because, in this case, a plane projective transformation exists between an object and the image plane; plane projective geometry provides an ideal mathematical tool for describing the transformation. As for general geometric configurations in three dimensions, Burns–Wiess–Riseman [3], [18] and Moses–Ullman [17] proved that we cannot calculate any invariant from a single view; we require at least two views. Two ways are then possible to derive invariants for noncoplanar objects: using two or more images and using some knowledge about the observed object.

Studies that took the first way, i.e., a strategy using multiple images, have been mainly reported. Three invariants of six points in a general position were derived from two weakly calibrated images [13] and two invariants of four lines in a general position were derived from two weakly calibrated images [12], [13]. Furthermore, one invariant of a pair of conics on two planes was derived from two weakly calibrated images [22]. Here, two images are called *weakly* calibrated when the epipolar geometry or the fundamental matrix [8] of the two images is determined *a priori*. These invariants were derived based on the property that a set of points in 3-D can be reconstructed up to a collineation from the point correspondences of two weakly calibrated images [6], [14]. Quan [21] extended this result to the case of three uncalibrated images, showing that three invariants of six points in a general position can be derived in a closed form from three uncalibrated images. In deriving these invariants, however, we implicitly reconstruct 3-D information (up to a collineation) and deal with space projective invariants. In other words, these invariants are derived by way of reconstruction. Once 3-D information is reconstructed, we need not stick to invariants; for example, we can alternatively use the object-centered coordinates in the canonical coordinate system.

Except for [23] and [24], there are no reports on studies that take the second way, i.e., a strategy imposing some assumptions on a noncoplanar object. In [23] and [24], three invariants of normal vectors of six planes for a trihedral object[1] were derived from a single view.

[1]If all the vertices of an object are characterized as the intersection of only three planes, it is called a

This paper is a study on projective invariants, derived from a single view, of noncoplanar objects on which some assumptions are imposed. It is shown that two projective invariants are derived from noncoplanar lines in a single view: one from five lines on two planes and the other from six lines on three planes. Moreover, conditions for nonsingularity, i.e., well-definedness and nondegeneracy, of the invariants are also given. Satisfying these conditions ensures that the values of the invariants are numerically stable when they are calculated in practical situations. In addition, some experimental results with real images are presented; we find that the values of the invariants over a number of viewpoints remain stable even for noisy images. Since the set of polygons with two planes (five lines are assumed to exist) includes the set of trihedrons (six planes are assumed to exist), the five-line-invariant derived in this paper can be applied to more general 3-D objects than the three invariants in [23] and [24]. When we compare the six-line-invariant with the five-line-invariant, while the number of lines increases, the assumptions imposed on the configuration of the lines are relaxed. We can then conclude that the six-line-invariant is more generally applicable than the five-line-invariant.

The outline of this paper is as follows. In Section 2, in preparation for further investigation, we introduce a projective framework. We then describe a property of three lines in the image plane, which we focus on in deriving invariants. In Section 3, we consider the representation of a line in 3-D: a pair of two planes. We also describe a motion in the projective framework. Here the motion for a line is assumed to be described by a projective general linear transformation in harmony with the projective framework. In Section 4, we present the main results of this paper: the existence of two projective invariants of noncoplanar lines. One invariant is derived from five lines on two planes; the other is from six lines on three planes. Part of this work was also presented in [28]. We first show the existence of the five-line-invariant, then the existence of the six-line-invariant. Both are based on the same properties of plane parameters whose proofs are postponed until the third subsection. In Section 5, necessary and sufficient conditions for nonsingularity of the invariants are investigated and they are given as Theorems 5.1 and 5.2. Some experimental results with real images are presented in Section 6. In this paper, we assume that an object moves around a fixed viewpoint and that the correspondence of lines among images is known. We also assume that readers are familiar with elementary projective

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trihedral object.

geometry, which can be found in [4], [7] or [26].

# 2 Three lines in the image plane

## 2.1 Camera model

We embed an object space in $\mathcal{P}^3$, the 3-dimensional projective space over the real number field, so that the Euclidean coordinates[2] $\tilde{\boldsymbol{x}} = (x, y, z)^{\mathrm{T}}$ of a point in 3-D are expressed by the homogeneous coordinates $\boldsymbol{x} = (x, y, z, 1)^{\mathrm{T}}$. In a similar way, we embed the image plane in $\mathcal{P}^2$, the projective plane over the real number field. If we assume a perspective projection (see Fig. 1) as the camera model, the camera then performs the projection from $\mathcal{P}^3$ to $\mathcal{P}^2$. This projection can be represented as a $3 \times 4$ matrix of rank three whose kernel is the homogeneous coordinates of the projection center, i.e., those of the viewpoint. Without loss of generality, we may take a coordinate system where the origin $\mathrm{O} = (0, 0, 0, 1)^{\mathrm{T}}$ is the projection center.

Let the homogeneous coordinates $\boldsymbol{x}$ of a point in three dimensions be projected to $\boldsymbol{X}$ in the image plane. We then have

$$\boldsymbol{X} = \lambda F_P \boldsymbol{x},$$

where $F_P$ is a $3 \times 4$ matrix of rank three and $\lambda$ is a nonzero real number. Since we take the coordinate system where the origin is the projection center, $F_P$ is represented as $F_P = (P \mid \mathbf{0})$ with a nonsingular $3 \times 3$ matrix $P$. Note that, in this formulation, all the information of the camera parameters is included in $P$; we need not assume that the camera is calibrated. It should also be remarked that introducing the projective space permits a compact representation of all changes of homogeneous coordinates as $4 \times 4$ matrices instead of as rotation matrices and translation vectors. This is because such changes are special cases of projective general linear transformations.

## 2.2 Three coplanar lines

For a line

$$aX + bY + c = 0$$

[2]We use a column vector.

(where $a^2 + b^2 \neq 0$) in the image plane embedded in $\mathcal{P}^2$, we obtain a vector $(a, b, c)^{\mathrm{T}}$. This vector is the homogeneous coordinates of the line.

The following fact is widely known for three different coplanar lines. To derive invariants, we focus on the value of the left-hand side of (2.1), which represents the volume of a parallelepiped (in three dimensions) constructed by the vectors representing the homogeneous coordinates of the three lines in the image plane.

**Observation 2.1** Let three different lines $i$ ($i = 1, 2, 3$) on the $XY-$plane be

$$a_i X + b_i Y + c_i \;=\; 0$$

(where $a_i^2 + b_i^2 \neq 0$). Then, they do not share a common point iff

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \;\neq\; 0. \tag{2.1}$$

□

**Remark 2.1** We can only determine $(a_i, b_i, c_i)^{\mathrm{T}}$ up to a scaling factor when we observe a line in the image plane. However, we can eliminate this indeterminacy by setting a criterion such as $a_i = 1$ or the normalization of the vector. □

# 3 Line representation and motion

A line in 3-D going through the origin (the viewpoint) makes a point in the image plane. In this paper, we assume that a line in 3-D does not go through the origin. In other words, a line in 3-D is assumed to be perspectively projected to a line in the image plane[3]. Such a line is called a line *in a general position.*

A line in a general position in 3-D is uniquely determined as a pair of planes, each of which never goes through the origin (see Fig. 2). Therefore, we represent a line in 3-D as a pair of planes. For a line (in 3-D) determined by planes $i$ and $j$, we denote by $\boldsymbol{N}_{ij}$ the homogeneous coordinates of the projected line in the image plane. For a line in the image plane, we call its homogeneous coordinates the *interpretation vector* for the line.

[3]A line on $z = 0$ is perspectively projected to a line at infinity in the image plane. This line is projectively not different from the other ones in the image plane.

Let a point (with the homogeneous coordinates $\boldsymbol{x}$) change its coordinates to $\boldsymbol{x}'$ after a motion. Here, projective general linear transformations of degree three are assumed to be admissible. Therefore,

$$\boldsymbol{x}' = \nu T \boldsymbol{x} \tag{3.1}$$

where $T \in \mathrm{PGL}(3)$ and $\nu$ is a nonzero real number. Note that PGL(3) denotes the projective general linear group of degree 3 over the real number field. It is important to remark that any rigid motion, i.e., a rotation around the viewpoint followed by a translation, is a special case of a projective general linear transformation. To be more precise, a rigid motion is expressed in the special form of

$$\left( \begin{array}{ccc|c} & \mathrm{R} & & \boldsymbol{t} \\ \hline 0 & 0 & 0 & 1 \end{array} \right),$$

where $R$ is a rotation matrix and $\boldsymbol{t}$ is a translation vector.

**Remark 3.1** There is indeterminacy of a scaling factor between the interpretation vector we actually obtain as a result of observing a line and the homogeneous coordinates of the projected line. □

## 4 Invariants of noncoplanar lines

In this section, the main results of this paper are presented as two theorems: Theorems 4.1 and 4.2. These two theorems ensure the existence of two invariants of noncoplanar lines, which is presented as Corollaries 4.1 and 4.2. First, the invariant of five lines on two planes is derived; next, the invariant of six lines on three planes is derived.

We assume that four different planes $i, j, k$ and $\ell$ ($i, j, k, \ell$ are natural numbers) in 3-D are given and that three lines are observed in the image plane, all of which are the images of the intersection lines of proper pairs of the four planes. We then consider three interpretation vectors, $\boldsymbol{N}_{ij}, \boldsymbol{N}_{jk}, \boldsymbol{N}_{k\ell}$ and define a $3 \times 3$ matrix $N_{ijk\ell}$ whose columns are these three vectors:

$$N_{ijk\ell} := [\, \boldsymbol{N}_{ij} \mid \boldsymbol{N}_{jk} \mid \boldsymbol{N}_{k\ell} \,].$$

In a similar manner, we define $\boldsymbol{N}'_{ij}$ and $N'_{ijkl}$ after any motion occurs.

## 4.1 Invariant of five lines on two planes

When six different planes, 1, 2, 3, 4, 5 and 6, are given, we can define $N_{1234}, N_{1235}, N_{6234}$ and $N_{6235}$. We then obtain the following theorem.

**Theorem 4.1** Let $\mathrm{rank}N_{i23j} = 3$ $(i = 1, 6\,;\, j = 4, 5)$. Then

$$\begin{aligned} \mathrm{rank}N'_{i23j} &= 3. && (4.1) \\ \frac{\det N_{1234}\cdot\det N_{6235}}{\det N_{1235}\cdot\det N_{6234}} &= \frac{\det N'_{1234}\cdot\det N'_{6235}}{\det N'_{1235}\cdot\det N'_{6234}}. && (4.2) \end{aligned}$$

*Proof*: As stressed before, for a line determined by planes $i$ and $j$, there is a scaling indeterminacy between the homogeneous coordinates $\boldsymbol{n}_{ij}$ of the projected line and the interpretation vector $\boldsymbol{N}_{ij}$ (see Remark 3.1). Hence

$$\boldsymbol{n}_{ij} = \rho_{ij}\boldsymbol{N}_{ij}, \tag{4.3}$$

where $\rho_{ij}$ is a nonzero real number. Here $\rho_{ij}$ is a scaling factor whose value is not known. Define $M_{ijk\ell}$ $(i, j, k, \ell$ are natural numbers) as a counterpart of $N_{ijk\ell}$:

$$M_{ijk\ell} := [\,\boldsymbol{n}_{ij} \mid \boldsymbol{n}_{jk} \mid \boldsymbol{n}_{k\ell}\,]. \tag{4.4}$$

(4.3) and (4.4) yield

$$\det M_{ijk\ell} = \rho_{ij}\cdot\rho_{jk}\cdot\rho_{k\ell}\cdot\det N_{ijk\ell}. \tag{4.5}$$

Similarly we define $\boldsymbol{n}'_{ij}$ and $M'_{ijkl}$ after a motion. Note that $\boldsymbol{n}'_{ij} = \rho'_{ij}\boldsymbol{N}'_{ij}$ $(\rho'_{ij} \neq 0)$, where the value of $\rho'_{ij}$ is unknown.

Definitions of $N_{ijk\ell}$ and $M_{ijk\ell}$ lead to $\mathrm{rank}N_{ijkl} = \mathrm{rank}M_{ijkl}$. Similarly, $\mathrm{rank}N'_{ijkl} = \mathrm{rank}M'_{ijkl}$. These yield (4.1) from Lemma 4.1 (2) below.

From (4.5) we obtain[4]

$$\text{LHS of (4.2)} = \frac{\det M_{1234}\cdot\det M_{6235}}{\det M_{1235}\cdot\det M_{6234}}.$$

On the other hand, it is easy to see

$$\text{RHS of (4.2)} = \frac{\det M'_{1234}\cdot\det M'_{6235}}{\det M'_{1235}\cdot\det M'_{6234}}.$$

Hence (4.2) is a consequence by Lemma 4.1 (1). □

[4]LHS and RHS mean the left-hand side and the right-hand side, respectively.

**Lemma 4.1**

(1) For $M_{1234}, M_{1235}, M'_{1234}$ and $M'_{1235}$,

$$\sigma_5 \cdot \det M'_{1234} \det M_{1235} \;=\; \sigma_4 \cdot \det M_{1234} \det M'_{1235}, \tag{4.6}$$

where $\sigma_4$ and $\sigma_5$ are nonzero real numbers.

(2) Let $\mathrm{rank} M_{ijk\ell} = 3$. Then

$$\mathrm{rank} M'_{ijk\ell} \;=\; 3.$$

*Proof*: The proof of this lemma will be given in Section 4.3. □

Theorem 4.1 shows that there exists a projective invariant for five lines,

$$I_5 \;:=\; \frac{\det N_{1234} \cdot \det N_{6235}}{\det N_{1235} \cdot \det N_{6234}},$$

all of which are characterized as the intersections of proper pairs of six planes. It is easy to see that the value of invariant $I_5$ generically depends on the five lines chosen. This indicates that an object generically has its own value of invariant $I_5$.

$I_5$ is calculated from the following five lines:

- $L_{12}$ (the intersection line of planes 1 and 2),
- $L_{23}$ (the intersection line of planes 2 and 3),
- $L_{34}$ (the intersection line of planes 3 and 4),
- $L_{35}$ (the intersection line of planes 3 and 5),
- $L_{62}$ (the intersection line of planes 6 and 2).

The configuration of these five lines in 3-D is characterized as follows:

1. The five lines are all on plane 2 or plane 3;
2. The intersection line of planes 2 and 3, $L_{23}$, is included;
3. There are two lines, $L_{12}$ and $L_{62}$, on plane 2 in addition to $L_{23}$;
4. There are two lines, $L_{34}$ and $L_{35}$, on plane 3 in addition to $L_{23}$.

**Corollary 4.1** There exists a projective invariant

$$I_5 \;:=\; \frac{\det N_{1234} \cdot \det N_{6235}}{\det N_{1235} \cdot \det N_{6234}} \tag{4.7}$$

for five lines on two planes (planes 2 and 3 above). The five lines include the intersection line of the two planes and two other lines on each plane (see Fig. 3). □

## 4.2 Invariant of six lines on three planes

When seven different planes, $1, 2, 3, 4, 5, 6$ and $7$, are given, we can define $N_{1234}, N_{1235}, N_{7634}$ and $N_{7635}$. We then obtain the following theorem.

**Theorem 4.2** Let $\mathrm{rank} N_{ij3k} = 3$ $(i = 1, 7\,;\, j = 2, 6\,;\, k = 4, 5)$. Then

$$\mathrm{rank} N'_{ij3k} \;=\; 3. \tag{4.8}$$

$$\frac{\det N_{1234} \cdot \det N_{7635}}{\det N_{1235} \cdot \det N_{7634}} \;=\; \frac{\det N'_{1234} \cdot \det N'_{7635}}{\det N'_{1235} \cdot \det N'_{7634}}. \tag{4.9}$$

*Proof*: (4.8) is now obvious due to Lemma 4.1 (2). It is easy to see

$$\text{LHS of (4.9)} \;=\; \frac{\det M_{1234} \cdot \det M_{7635}}{\det M_{1235} \cdot \det M_{7634}},$$

$$\text{RHS of (4.9)} \;=\; \frac{\det M'_{1234} \cdot \det M'_{7635}}{\det M'_{1235} \cdot \det M'_{7634}}.$$

Therefore, (4.9) is a consequence by Lemma 4.1 (1). □

Theorem 4.2 states that there exists a projective invariant for six lines,

$$I_6 \;:=\; \frac{\det N_{1234} \cdot \det N_{7635}}{\det N_{1235} \cdot \det N_{7634}},$$

all of which are the intersections of proper pairs of seven planes. It should again be noted that the value of invariant $I_6$ generically depends on the six lines chosen; an object, therefore, generically has its own value of invariant $I_6$.

$I_6$ is calculated from the following six lines:

- $L_{12}$ (the intersection line of planes 1 and 2),
- $L_{23}$ (the intersection line of planes 2 and 3),
- $L_{34}$ (the intersection line of planes 3 and 4),

- $L_{35}$ (the intersection line of planes 3 and 5),
- $L_{76}$ (the intersection line of planes 7 and 6),
- $L_{63}$ (the intersection line of planes 6 and 3).

The configuration of these six lines in 3-D is characterized as follows:

1. the six lines are all on plane 2, 3 or 6;
2. the intersection line, $L_{23}$, of planes 2 and 3 is included;
3. the intersection line, $L_{63}$, of planes 6 and 3 is included;
4. there are two lines, $L_{34}$ and $L_{35}$, on plane 3 in addition to $L_{23}$ and $L_{63}$;
5. there is one line, $L_{12}$, on plane 2 in addition to $L_{23}$;
6. there is one line, $L_{76}$, on plane 6 in addition to $L_{63}$.

**Corollary 4.2** There exists a projective invariant

$$I_6 \;:=\; \frac{\det N_{1234} \cdot \det N_{7635}}{\det N_{1235} \cdot \det N_{7634}} \tag{4.10}$$

for six lines on three planes (planes 2, 3 and 6 above). For three aligned planes, the six lines include: (i) the two intersection lines of the adjacent planes in the alignment; (ii) two other lines on the middle plane; and (iii) one other line on each side plane (see Fig. 4). □

## 4.3 Proof of Lemma 4.1

Here, we give the proof of Lemma 4.1. In the first subsection, we introduce the interpretation plane for a line in the image plane and then clarify a relationship between the homogeneous coordinates of the interpretation plane and the interpretation vector for the line. In the second subsection, we turn to the proof of Lemma 4.1 based on the relationship.

### 4.3.1 Interpretation plane for a line

For a line in the image plane, we consider the plane on which both the origin (the viewpoint) and the line exist (see Fig. 5). We refer to this plane as the *interpretation plane* for the line. It is easy to see that, for a line in the image plane, any line in 3-D that exists in its interpretation plane is projected to the line (in the image plane). It should be remarked that we use the interpretation plane *of* (instead of "for") a line in the case where the line is not in the image plane but in 3-D.

For a line (in the image plane), its interpretation vector is obtained as a result of applying[5] $F_{P^{-\mathrm{T}}}$ to the homogeneous coordinates of the interpretation plane for the line. This can be understood in the following way. Namely, for a line

$$aX + bY + c \;=\; 0 \tag{4.11}$$

(where $a^2 + b^2 \neq 0$) in the image plane, let $\boldsymbol{X}$ ($\boldsymbol{X} \neq \boldsymbol{0}$) be the homogeneous coordinates of any point in the line and put $\tilde{\boldsymbol{X}} = P^{-1}\boldsymbol{X} \left(= (\tilde{X}, \tilde{Y}, \tilde{Z})^{\mathrm{T}}\right)$. Then, $(\tilde{X}, \tilde{Y}, \tilde{Z}, 1)^{\mathrm{T}}$ is an inverse image of $\boldsymbol{X}$ with respect to $F_P$. (In other words, a point whose homogeneous coordinates are $(\tilde{X}, \tilde{Y}, \tilde{Z}, 1)^{\mathrm{T}}$ is projected to the point (with the homogeneous coordinates $\boldsymbol{X}$) in the image plane by $F_P$.) Moreover, put $\boldsymbol{c} = P^{\mathrm{T}}(a, b, c)^{\mathrm{T}} \left(= (\tilde{a}, \tilde{b}, \tilde{c})^{\mathrm{T}}\right)$, then (4.11) is rewritten as

$$(\tilde{a}, \tilde{b}, \tilde{c}, 0)^{\mathrm{T}} \cdot (\tilde{X}, \tilde{Y}, \tilde{Z}, 1)^{\mathrm{T}} \;=\; 0. \tag{4.12}$$

(4.12) represents the plane on which both the origin O and line (4.11) exist. Hence, (4.12) is the interpretation plane for line (4.11); $(\tilde{a}, \tilde{b}, \tilde{c}, 0)^{\mathrm{T}}$ is the homogeneous coordinates of the interpretation plane. From $(a, b, c)^{\mathrm{T}} = P^{-\mathrm{T}}\boldsymbol{c} = F_{P^{-\mathrm{T}}}\,(\tilde{a}, \tilde{b}, \tilde{c}, 0)^{\mathrm{T}}$, we can see that interpretation vector $(a, b, c)^{\mathrm{T}}$ is obtained by applying $F_{P^{-\mathrm{T}}}$ to the homogeneous coordinates of the interpretation plane for line (4.11).

As seen before, we represent a line as a pair of planes, each of which never goes through the origin. Thus, for an intersection line of two planes, we next consider the relationship between the interpretation vector for the projected line and the homogeneous coordinates of the two planes. Let two planes $i$ ($i = 1, 2$) in 3-D be

$$a_i x + b_i y + c_i z + d_i \;=\; 0 \tag{4.13}$$

(where $d_i \cdot (a_i^2 + b_i^2 + c_i^2) \neq 0$). Denote by $\boldsymbol{n}_i$ the homogeneous coordinates of plane $i$, namely,

$$\boldsymbol{n}_i \;=\; (a_i, b_i, c_i, d_i)^{\mathrm{T}},$$

[5]For a square matrix $P$, $P^{-\mathrm{T}}$ is $(P^{\mathrm{T}})^{-1}$ or equivalently $(P^{-1})^{\mathrm{T}}$.

then (4.13) is rewritten as

$$n_i \cdot x \;=\; 0, \tag{4.14}$$

where $x = (x, y, z, 1)^{\mathrm{T}}$. Hence, $x$, the homogeneous coordinates of a point that is on both planes 1 and 2, satisfies

$$\sum_{i=1}^{2} \mu_i \left( n_i \cdot x \right) \;=\; 0, \tag{4.15}$$

where $\mu_i$ $(i = 1, 2)$ are real numbers. By fixing the values of $\mu_i$ $(i = 1, 2)$ so that the coordinates of the origin O satisfy (4.15), we obtain the interpretation plane of the intersection line of planes 1 and 2:

$$(d_2 n_1 - d_1 n_2) \cdot x \;=\; 0.$$

Therefore, $d_2 n_1 - d_1 n_2$ is the homogeneous coordinates of the interpretation plane of the intersection line of planes 1 and 2; we obtain $F_{P^{-\mathrm{T}}}(d_2 n_1 - d_1 n_2)$ when we observe the line in 3-D determined by $n_i$'s $(i = 1, 2)$.

**Remark 4.1** If we set $d_i = 0$ in (4.13), then all the lines in plane $i$ are observed as the same line in the image plane. This shows that the interpretation vectors for their projected lines coincide. Throughout this paper, we assume that no two different lines are observed to be coincident. □

### 4.3.2 Proof

We will now prove Lemma 4.1. To prove Lemma 4.1 (1), it suffices to show equation (4.16) below. This is because, when four different planes are $1, 2, 3$ and 5, we have the following equation as a counterpart of (4.16):

$$\det T \cdot d_2 d_3 \cdot \det M'_{1235} \;=\; \sigma_1 \sigma_2 \sigma_3 \sigma_5 \cdot d'_2 d'_3 \cdot \det M_{1235};$$

this is then combined with (4.16) to obtain (4.6) (see Remark 4.2).

$$\det T \cdot d_2 d_3 \cdot \det M'_{1234} \;=\; \prod_{i=1}^{4} \sigma_i \cdot d'_2 d'_3 \cdot \det M_{1234}. \tag{4.16}$$

Let $D_{1234} := [\, n_1 \mid n_2 \mid n_3 \mid n_4 \,]$. We then obtain

$$\det M_{1234} \;=\; \frac{1}{\det P} \cdot d_2 d_3 \cdot \det D_{1234}. \tag{4.17}$$

Similarly, define $D'_{1234} := [\ \boldsymbol{n}'_1 \mid \boldsymbol{n}'_2 \mid \boldsymbol{n}'_3 \mid \boldsymbol{n}'_4\ ]$ after a motion. When a point in 3-D is subject to a motion of (3.1), $\boldsymbol{n}_i$, plane parameters of (4.14), is subject to

$$\boldsymbol{n}'_i \;=\; \sigma_i T^{-\mathrm{T}} \boldsymbol{n}_i, \tag{4.18}$$

where $\sigma_i\,(\,i = 1, 2)$ are nonzero real numbers. We then have

$$\begin{aligned} \det M'_{1234} &= \frac{1}{\det P} \cdot d'_2 d'_3 \cdot \det D'_{1234} \\ &= \frac{1}{\det P} \cdot \frac{1}{\det T} \cdot \prod_{i=1}^{4} \sigma_i \cdot d'_2 d'_3 \cdot \det D_{1234} \end{aligned} \tag{4.19}$$

since (4.18) yields

$$\det D'_{ijk} \;=\; \frac{1}{\det T} \cdot \prod_{i=1}^{4} \sigma_i \cdot \det D_{ijk}.$$

(4.17) and (4.19) immediately yield (4.16) (see Remark 4.2). It should be noted that (4.16) is independent of $F_P$, the projection from $\mathcal{P}^3$ to $\mathcal{P}^2$.

We now turn to the proof of Lemma 4.1 (2). It is easy to see that (4.20) is equivalent to Lemma 4.1 (2).

$$\mathrm{rank} M_{1234} = 3 \;\Longrightarrow\; \mathrm{rank} M'_{1234} = 3. \tag{4.20}$$

Since $d_2 d_3 \neq 0$, it follows from (4.17) that $\mathrm{rank} M_{1234} = 3$ is equivalent to $\mathrm{rank} D_{1234} = 4$. Similarly, $\mathrm{rank} M'_{1234} = 3$ is equivalent to $\mathrm{rank} D_{1234} = 4$ from (4.19) since $\sigma_i \neq 0\,(i = 1, 2, 3, 4)$ and $d'_2 d'_3 \neq 0$. This completes the proof of (4.20).

**Remark 4.2** If $d'_2 = 0$, then both the intersection line of planes 1 and 2, and that of planes 2 and 3, are observed to be coincident after a motion (see Remark 4.1). On the other hand, if $d'_3 = 0$, then both the intersection line of planes 2 and 3, and that of planes 3 and 4, are observed to be coincident. These facts show that if $d'_2 \cdot d'_3 = 0$, then the number of visible lines changes before and after a motion. In this paper, we do not assume that such a change occurs, which leads to $d'_2 \cdot d'_3 \neq 0$. □

**Remark 4.3** (4.17) and (4.19) show that $\det M'_{1234} = 0$ is equivalent to $\det M_{1234} = 0$. Hence, $\det N'_{1234} = 0$ is equivalent to $\det N_{1234} = 0$ since $\mathrm{rank} M_{1234} = \mathrm{rank} N_{1234}$ and $\mathrm{rank} M'_{1234} = \mathrm{rank} N'_{1234}$ (see the proof of Theorem 4.1). Namely, once three lines do not share a common point in the image plane, we can guarantee that these three lines after any motion never share a common point in the image plane. □

## 5 Nonsingularity conditions

In this section, we give the necessary and sufficient conditions under which the invariants derived in the previous section are nonsingular: the nonsingularity conditions for $I_5$ and $I_6$. Here, we define "an invariant is nonsingular" as "the value of the invariant is not 0, $\infty$ or $^0/_0$". Nonsingularity can be regarded as nondegeneracy and well-definedness. As we can see, the nonsingularity conditions for an invariant ensure that its values are numerically stable when they are calculated in practical situations.

It is easy to see that $I_5$ is nonsingular iff the values of the determinants of $N_{ijk\ell}$'s in (4.7) are not zero; $I_6$ is nonsingular iff the values of the determinants of $N_{ijk\ell}$'s in (4.10) are not zero. Let three lines in 3-D, i.e., the intersection line of planes $i$ and $j$, that of $j$ and $k$, and that of $k$ and $\ell$, satisfy one of the following:

(i) they share a common point in 3-D,

(ii) they are parallel with each other and are not parallel to the image plane.

These three lines then share a common point in the image plane through the perspective projection. Hence, $\det N_{ijk\ell}$ is equal to zero (see Observation 2.1).

**Theorem 5.1** The necessary and sufficient condition under which $I_5$ is nonsingular is that the five lines on two planes have the following property:

> For three lines, i.e., the intersection line of the two planes and any two noncoplanar lines from among the other four (we have four cases), (I) or (II) is satisfied.
>
> (I) They are not parallel with each other and never share a common point in 3-D.
>
> (II) They are parallel with each other and are parallel to the image plane. □

**Theorem 5.2** For six lines from whose configuration we can calculate $I_6$, let $A$ be the plane where four lines among the six exist and $B$, $C$ be the other planes. The necessary and sufficient condition under which $I_6$ is nonsingular is that the six lines on three planes $A$, $B$ and $C$ have the following property:

> For three lines, i.e., the two lines of plane $B$ [$C$] and any one of the lines on plane $A$ that is neither the intersection of $A$ and $B$ nor that of $A$ and $C$ (we have four cases), (I) or (II) is satisfied.

(I) They are not parallel with each other and never share a common point in 3-D.

(II) They are parallel with each other and are parallel to the image plane. □

We can derive four collinear points if five lines exist on two planes and they include the intersection line of the two planes: extension of the four other lines on the planes to the intersection line of the two planes makes four collinear points. We can then calculate the cross ratio of the four points, which is unaffected by a change in viewpoint. However, when some of the four lines are parallel to the intersection line (e.g., Fig. 6), we cannot derive four collinear points; we cannot calculate the cross ratio. In contrast to this, we can calculate the invariants, $I_5$ and $I_6$, which are nonsingular even for such a case. Therefore, the invariants $I_5$ and $I_6$ are more generally applicable than the four extracted collinear points; $I_5$ and $I_6$ can be applied even to situations in which an invariant cannot be extracted directly from a simple cross ratio.

**Remark 5.1** When we observe six points on two planes such that

- there are two points on the intersection line of the two planes and,
- there are two other points on each plane,

it is easy to see that we can construct five lines on two planes, from which we can calculate invariant $I_5$ and that satisfy the condition for nonsingularity (see Fig. 7). This shows that the same invariant exists for six points on two planes. □

**Remark 5.2** When we observe seven points on three aligned planes such that

- there are two points on each intersection line of the two adjacent planes in the alignment and,
- there is one other point on each plane,

it is easy to see that we can construct six lines on three planes, from which we can calculate invariant $I_6$ and that satisfy the condition for nonsingularity (see Fig. 8). This shows that the same invariant exists for seven points on three planes. □

## 6 Experimental results

In Section 4, we proved the existence of two projective invariants, $I_5$ and $I_6$. $I_5$ is derived from five lines on two planes; the five lines include the intersection line of the two planes and two other lines on each plane. Whereas, $I_6$ is derived from six lines on three planes; when the three planes are aligned, these six lines include the two intersection lines of the adjacent planes in the alignment, two other lines on the middle plane and one other line on each side plane. Furthermore, in Section 5, we gave the nonsingularity conditions to the two invariants as Theorems 5.1 and 5.2. On the basis of these results, our experimental results with real images are shown.

Two objects, polygons I and II (Figs. 9 and 10), are used to calculate values of the invariants, $I_5$ and $I_6$. These two objects are constructed from a parallelepiped or a rectangular parallelepiped on which the same triangular prism is attached. We can assume that they are similar to each other.

These polygons were randomly moved by hand. We obtained several images[6] of polygons I and II by using a fixed camera that was not calibrated. For each image, we first applied a low pass filter of a $3 \times 3$ weighted kernel window to reduce noise. We then calculated the Laplacian with an 8-neighbor weighted coefficient matrix to extract the edges (Figs. 12 and 13). Next, to each edge in the image, we applied the method of least squares to find the equation of the line that represents the edge. We also attached labels to the planes and the edges of the polygons as shown in Fig. 11.

**Calculation of $I_5$:** We chose (A) and (B) as two planes on which five lines should exist. We then selected five out of seven lines,[7] $1, 2, \ldots, 7$, on (A) or (B) that satisfy the nonsingularity condition for $I_5$ in order to calculate the values of $I_5$. There are nine combinations in selecting five lines out of the seven such that they include line 4, the intersection of (A) and (B), and two other lines on each of (A) and (B). However, we essentially have only four combinations that give independent values of $I_5$ (see the definition of $I_5$ in (4.7)). Thus, for the lines that were obtained from six edge images (a), ..., (f)

[6] Each image consists of $480 \times 512$ pixels, and each pixel is assigned a natural number from $0 \sim 255$ as the value of its grey level.

[7] Line $i$ means the line that represents edge $i$ ($i \in \{1, \ldots, 10\}$).

in Fig. 12, we calculated the values of these four invariants, which are shown in Table 1. We denote by $I_{ijklm}$ the invariant of five lines $i, j, k, l, m$ ($i, j, k, l, m \in \{1, 2, \ldots, 7\}$). For each combination of five lines, we also showed the mean $m$ over the six images, the standard deviation $\sigma$ and the percentage of the standard deviation of the mean. Similarly, the values for the six edge images in Fig. 13 are shown in Table 2.

Tables 1 and 2 show that all of the values of $I_{ijklm}$ are essentially constants: they remain stable in spite of a change in viewpoint. This shows that the values of $I_5$ are reliable even for noisy images. Furthermore, their values significantly depend on the object, even though the two objects are similar to each other. For each object, they also depend on the five lines chosen, i.e., a combination of the observed five lines. These show that each object generically has its own value of $I_5$.

**Calculation of $I_6$:** We chose (A), (B) and (C) as three planes on which six lines should exist. We then selected six out of ten lines, $1, 2, \ldots, 10$, on (A), (B) or (C) that satisfy the nonsingularity condition for $I_6$ in order to calculate the value of invariant $I_6$, as is the case of $I_5$. There are three combinations[8] in selecting these six lines. Thus, for the six edge images (a), ..., (f) in Fig. 12, we calculated the values of these three invariants, which are shown in Table 3. We denote by $I_{ijk\ell mn}$ the invariant of six lines $i, j, k, \ell, m, n$ ($i, j, k, \ell, m, n \in \{1, 2, \ldots, 10\}$). Note that we cannot derive four collinear points from six lines $2, 4, 5, 6, 7, 9$ since 2 and 4, 6 and 9 are respectively parallel in 3-D: $I_{245679}$ can never be obtained by means of a cross ratio. The same is true of $I_{145679}$ and $I_{345679}$. For each combination of six lines, we also showed the mean $m$ over the six images, the standard deviation $\sigma$ and the percentage of the standard deviation to the mean. Similarly, the values for the six edge images in Fig. 13 are shown in Table 4.

Tables 3 and 4 show that all of the values of $I_{ijk\ell mn}$ are essentially constants: they remain stable in spite of a change in viewpoint. This shows that the values of $I_6$ are also reliable even for noisy images. As is the case of $I_5$, we can see that each object generically also has its own value of $I_6$.

[8]Since three lines 5,6,8 and three lines 6,7,10 respectively share common points in 3-D, a combination where these three lines are included does not satisfy the nonsingularity condition for $I_6$.

As shown above, for a real 3-D object, we found that the values of $I_5$ as well as $I_6$ are respectively unaffected by a change in viewpoint and that the object has its own value of $I_5$ and that of $I_6$. Therefore, both $I_5$ and $I_6$ can be important in identifying one object out of many.

## 7 Conclusion

We proved the existence of two projective invariants $I_5$ and $I_6$. $I_5$ is derived from five lines on two planes; these five lines include the intersection line of the two planes and two other lines on each plane. Whereas, $I_6$ is derived from six lines on three planes; when the three planes are aligned, these six lines include the two intersection lines of the adjacent planes in the alignment, two other lines on the middle plane and one other line on each side plane. The five lines for $I_5$ and the six lines of $I_6$ exist three-dimensionally, respectively. Hence, invariants $I_5$ and $I_6$ are derived from noncoplanar lines, namely, 3-D objects.

Furthermore, the nonsingularity conditions for the two invariants, i.e., the necessary and sufficient conditions making them nonsingular, were also given. $I_5$ is nonsingular iff (I) or (II) below is satisfied by three lines among the five lines, i.e., the intersection line of the two planes and any two noncoplanar lines from among the other four (we have four cases). $I_6$ is nonsingular iff (I) or (II) is satisfied by three lines among the six lines, i.e., the two coplanar lines on either of the side planes and any one of the lines on the middle plane that is neither of the two intersection lines (we have four cases).

(I) They are not parallel with each other and never share a common point in 3-D.

(II) They are parallel with each other and are parallel to the image plane.

The nonsingularity conditions guarantee that the invariants are not only well-defined but nondegenerate; they also ensure that the values of the invariants are numerically stable when they are calculated in practical situations. It is important to remark here that we have no other functionally independent invariants that can be derived by the method used in this paper, which is shown in [27].

We applied these theoretical results to real images, and found that the values of the invariants remain stable even for noisy images and that an object generically has its own value of the invariants. These indicate that values of $I_5$ and $I_6$ are reliable and that they give important information about an object. Therefore, both $I_5$ and $I_6$ can be important in identifying one object out of many.

The values of $I_5$ depend on the order of the five lines in the computation (see (4.7)). The values of $I_6$ depend on the order of the six lines in the computation (see (4.10)). Namely, a different ordering of the lines, i.e., associating indices with the lines in a different way, can yield different values of $I_5$ and $I_6$. If the values of $I_5$ and $I_6$ are insensitive to the order, then we need not establish the line correspondence in a certain sense; we can avoid storing in a recognition system all the values for every possible ordering of the lines from which $I_5$ or $I_6$ is computed. To make them insensitive to the order, we should derive order-independent invariants from $I_5$ and $I_6$ respectively, such as $j$-invariant [19] in the case of four collinear points or $p^2$-invariant [16] in the case of five coplanar points. Since both $I_5$ and $I_6$ are in similar forms of cross ratio, it would be possible to derive order-independent invariants from them. Elaboration of deriving such invariants is left open for future research. Also left for future investigations are: 1) the theoretical analysis of the noise sensitivity of the invariants; and 2) the theoretical analysis of the case where the invariants are not nonsingular, so that we can use invariants even in such a case.

## Acknowledgments

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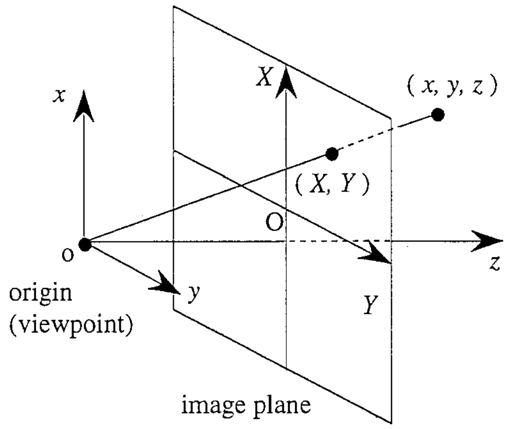



Fig. 1: Perspective projection centered at the origin

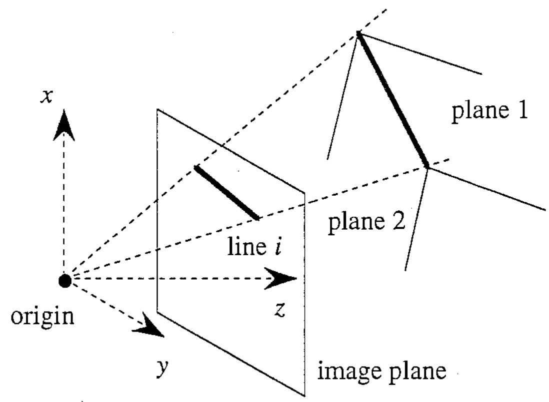



Fig. 2: Line determined by a pair of planes

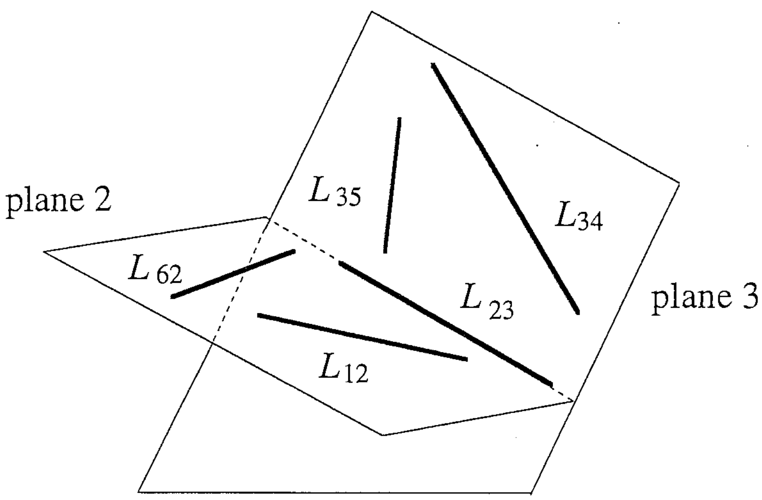



Fig. 3: Five lines on two planes

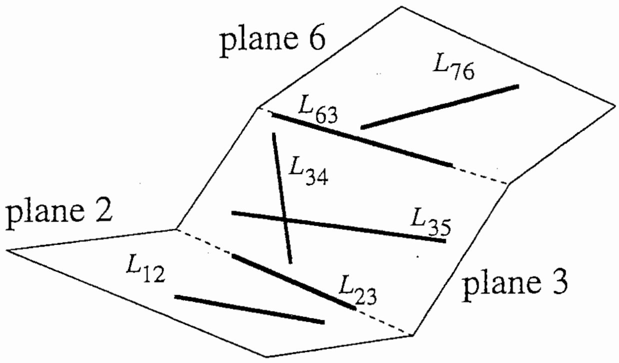



Fig. 4: Six lines on three planes

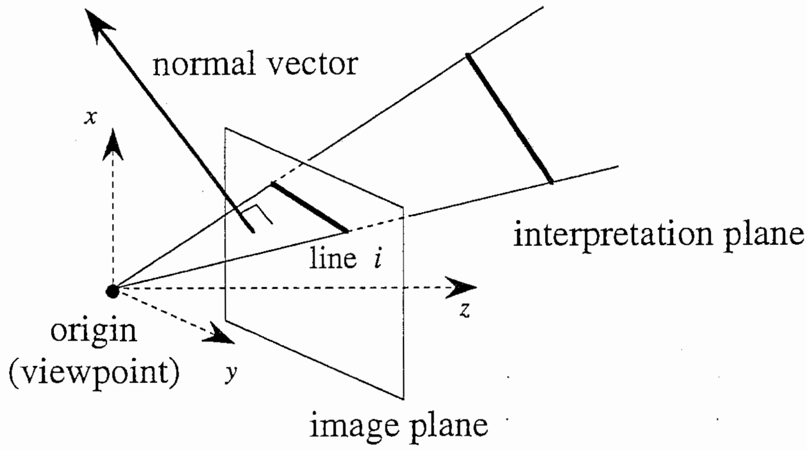



Fig. 5: Line $i$ and the homogeneous coordinates of the interpretation plane

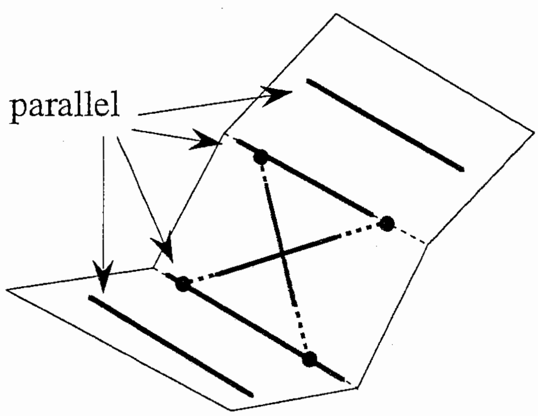



Fig. 6: Six lines that never make four collinear points and that satisfy the nonsingularity condition

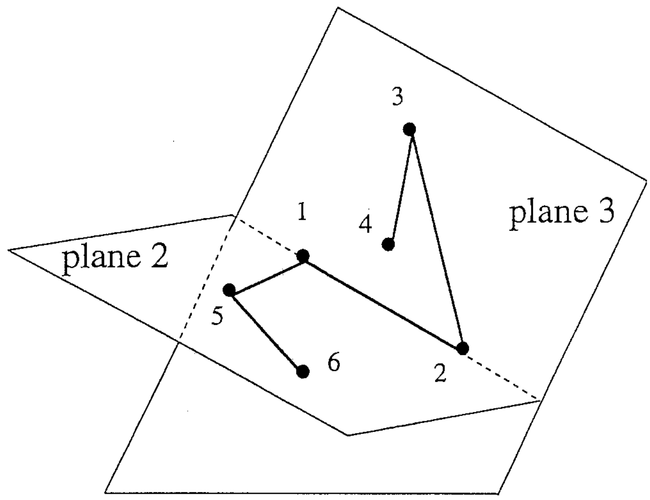



Fig. 7: Five lines derived from six points on two planes

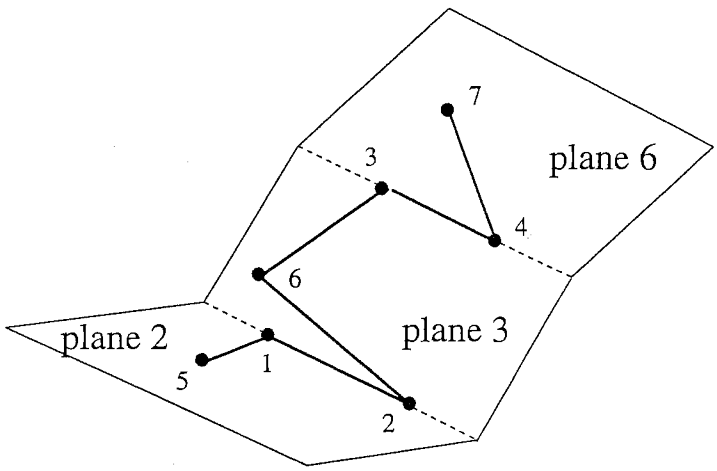



Fig. 8: Six lines derived from seven points on three planes

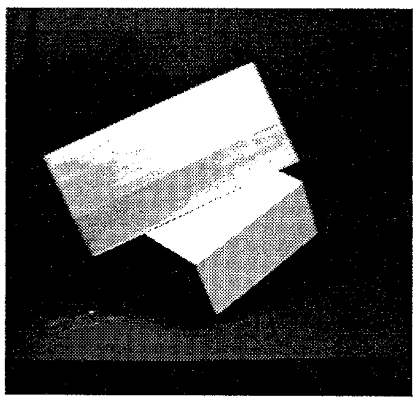


Fig. 9: Polygon I

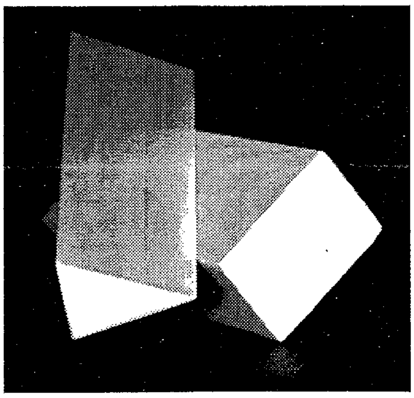


Fig. 10: Polygon II

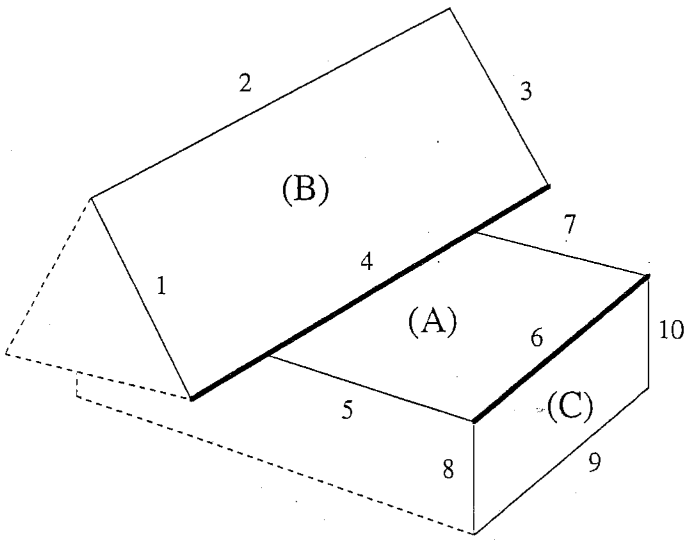



Fig. 11: Labels for planes and lines of polygons I and II

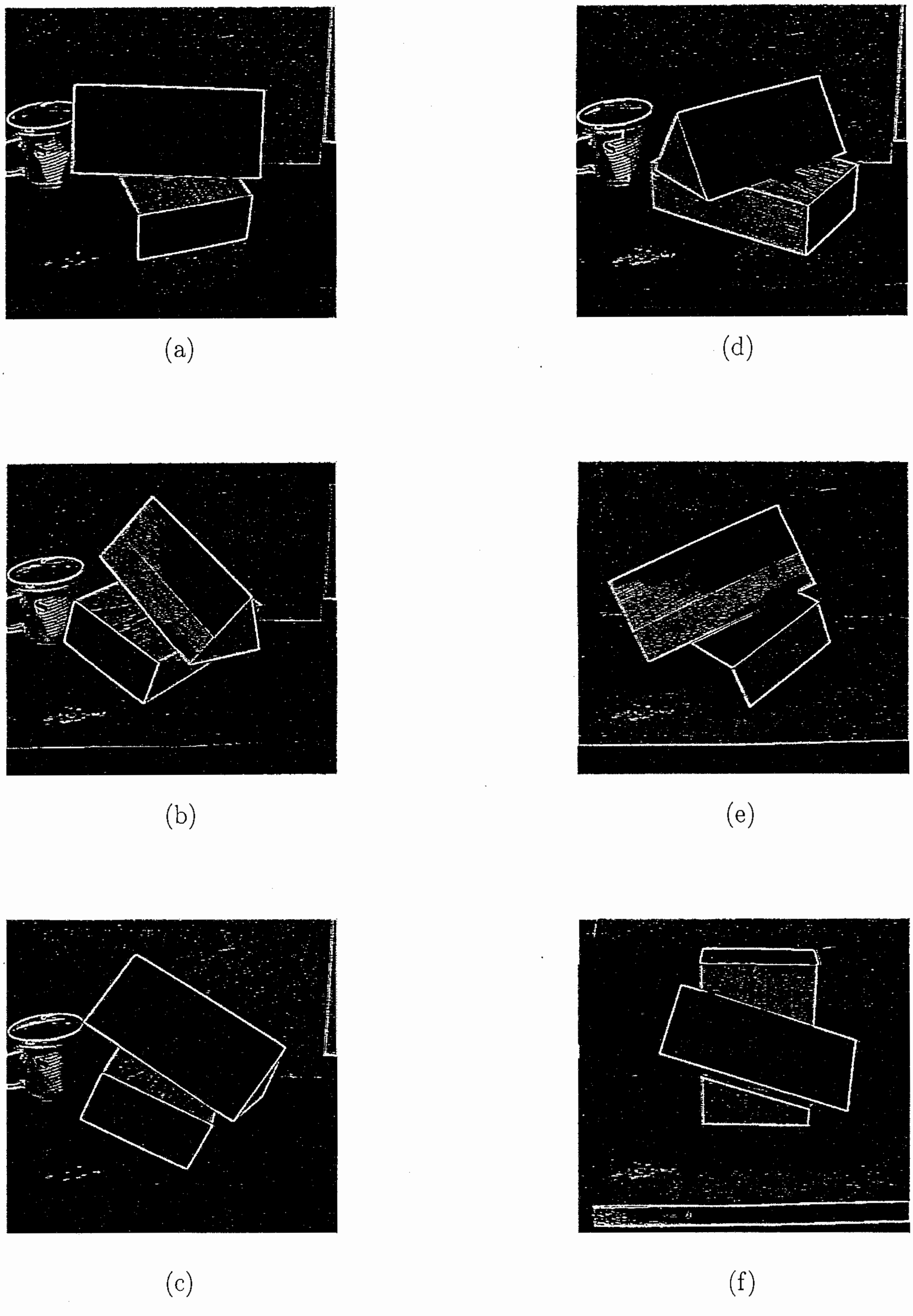
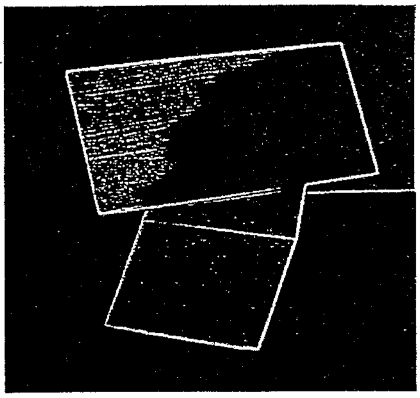
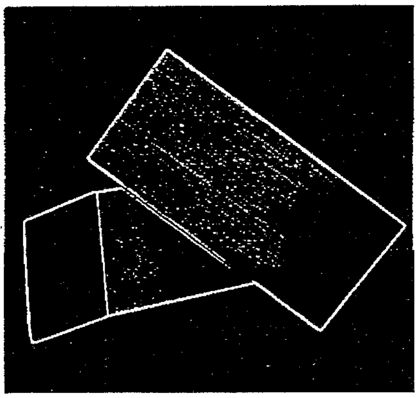


Fig. 12: Extracted edges from images of the polygon I

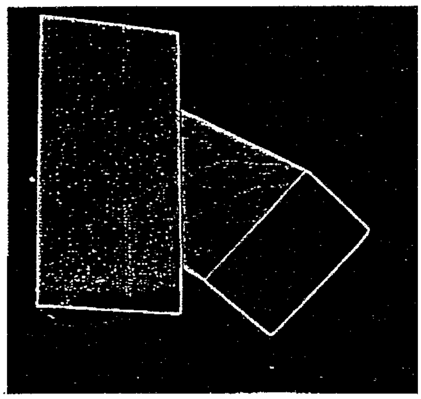
We obtained several images of the polygon in Fig. 9 using a fixed camera that is not calibrated. The polygon was moved randomly by hand. For each image, we first applied a low pass filter of a $3 \times 3$ weighted kernel window to reduce noise. We then calculated the Laplacian with an 8-neighbor weighted coefficient matrix to extract edges ((e) is the extracted edge image for Fig. 9).



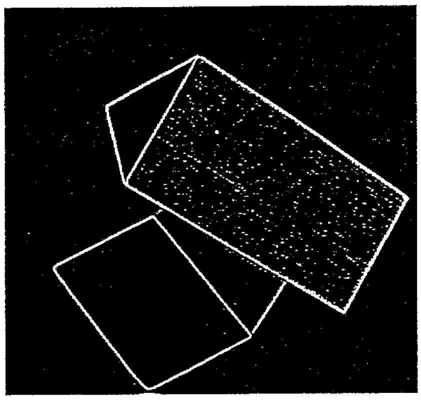
(a)



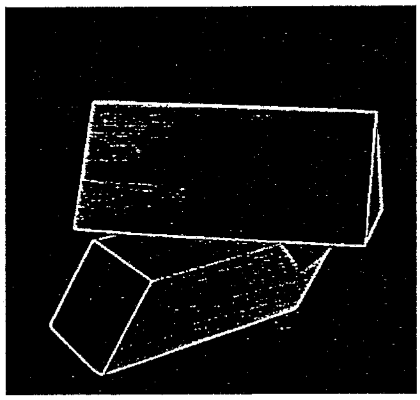
(b)



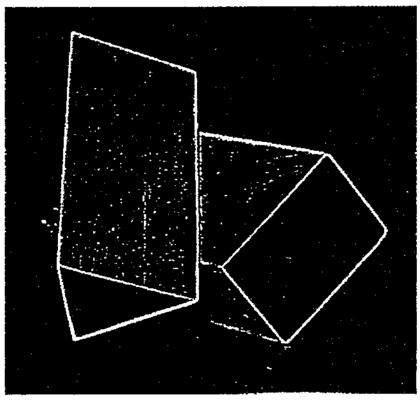
(c)



(d)



(e)



(f)

Fig. 13: Extracted edges from images of the polygon II

We obtained several images of the polygon in Fig. 10 using a fixed camera that is not calibrated. The polygon was moved randomly by hand. For each image, we first applied a low pass filter of a $3 \times 3$ weighted kernel window to reduce noise. We then calculated the Laplacian with an 8-neighbor weighted coefficient matrix to extract edges ((f) is the extracted edge image for Fig. 10).

Table 1: Values of invariant $I_5$ for polygon I

Shown are the values of $I_5$ calculated from the lines representing the edges in Fig. 12 Also shown are their means $m$, standard deviations $\sigma$ and the percentages of the standard deviations of the means.

| | $I_{13457}$ | $I_{13467}$ | $I_{23457}$ | $I_{23467}$ |
|---|---|---|---|---|
| (a) | 0.043451 | −0.46719 | 0.15229 | −0.30025 |
| (b) | 0.039659 | −0.39997 | 0.14043 | −0.25306 |
| (c) | 0.042730 | −0.45133 | 0.14967 | −0.28919 |
| (d) | 0.043441 | −0.43375 | 0.15562 | −0.26561 |
| (e) | 0.039769 | −0.39784 | 0.14500 | −0.24465 |
| (f) | 0.041425 | −0.46856 | 0.14702 | −0.30678 |
| $m$ | 0.041746 | −0.43644 | 0.14834 | −0.27659 |
| $\sigma$ | 0.001583 | 0.02894 | 0.004880 | 0.02354 |
| $\sigma/m$ (%) | 3.79 | 6.63 | 3.29 | 8.51 |

Table 2: Values of invariant $I_5$ for polygon II

Shown are the values of $I_5$ calculated from the lines representing the edges in Fig. 13. Also shown are their means $m$, standard deviations $\sigma$ and the percentages of the standard deviations of the means.

| | $I_{13457}$ | $I_{13467}$ | $I_{23457}$ | $I_{23467}$ |
|---|---|---|---|---|
| (a) | 0.077110 | 0.010698 | 0.33941 | 0.28157 |
| (b) | 0.078296 | 0.012650 | 0.33247 | 0.28493 |
| (c) | 0.078018 | 0.012668 | 0.34063 | 0.29389 |
| (d) | 0.075384 | 0.011608 | 0.33648 | 0.28712 |
| (e) | 0.079306 | 0.010014 | 0.32180 | 0.27076 |
| (f) | 0.073955 | 0.012426 | 0.34134 | 0.29828 |
| $m$ | 0.077012 | 0.011677 | 0.33535 | 0.28609 |
| $\sigma$ | 0.001823 | 0.001018 | 0.006747 | 0.008825 |
| $\sigma/m$ (%) | 2.37 | 8.72 | 2.01 | 3.08 |

Table 3: Values of invariant $I_6$ for polygon I

Shown are the values of $I_6$ calculated from the lines that represent the edges in Fig. 12. Also shown are their means $m$, standard deviations $\sigma$ and the percentages of the standard deviations to the means.

| | $I_{145679}$ | $I_{245679}$ | $I_{345679}$ |
|---|---|---|---|
| (a) | 0.283451 | 0.993436 | 6.523431 |
| (b) | 0.275090 | 0.974098 | 6.936328 |
| (c) | 0.287415 | 1.006754 | 6.726357 |
| (d) | 0.258875 | 0.927373 | 5.959299 |
| (e) | 0.269928 | 0.984172 | 6.787387 |
| (f) | 0.289249 | 1.026566 | 6.982462 |
| $m$ | 0.277335 | 0.985400 | 6.652544 |
| $\sigma$ | 0.010659 | 0.030852 | 0.344063 |
| $\sigma/m$ (%) | 3.84 | 3.13 | 5.17 |

Table 4: Values of invariant $I_6$ for polygon II

Shown are the values of $I_6$ calculated from the lines that represent the edges in Fig. 13. Also shown are their means $m$, standard deviations $\sigma$ and the percentages of the standard deviations to the means.

| | $I_{145679}$ | $I_{245679}$ | $I_{345679}$ |
|---|---|---|---|
| (a) | 0.226883 | 0.998656 | 2.942333 |
| (b) | 0.229346 | 0.973880 | 2.929213 |
| (c) | 0.224297 | 0.979282 | 2.874939 |
| (d) | 0.229489 | 1.024318 | 3.044252 |
| (e) | 0.244337 | 0.991458 | 3.080928 |
| (f) | 0.221639 | 1.022973 | 2.996958 |
| $m$ | 0.229332 | 0.998428 | 2.978104 |
| $\sigma$ | 0.007254 | 0.019535 | 0.070260 |
| $\sigma/m$ (%) | 3.16 | 1.96 | 2.36 |