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Why the 1/3 Power Law of Drawing and Planar Motion Perception?

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While drawing a plane curve, the instantaneous tangential velocity of the hand decreases as the curvature increases [1, 2, 3]. This relationship is best described as a power law where velocity is proportional to the 1/3 power of the radius of curvature [3]. An identical power law is observed in planar motion perception [4, 5]. Although aspects of drawing [6, 7, 8, 9], its development [10], as well as the visual perception of motion [4, 5], show that the power law influences the organization of both perception and action, no adequate explanation for the specific 1/3 value has been offered. Here we show by way of affine differential geometry and human drawing data that the 1/3 power law means that curves are drawn with constant affine velocity. This theoretical finding suggests that the 1/3 power law results from approximations in visuo-motor transformations involving affine rather than Euclidean distances.

Although the physical world which humans see and manipulate can be described by Euclidean geometry, there is reason to doubt that properties such as Euclidean distance and angles are faithfully reproduced in our internal representations. For example, judgments of static form show that the structure of human visual space [11, 12] as well as motor space [13] deviate from Euclidean geometry. Nonetheless, humans successfully interact with this Euclidean world and it can be questioned how these deviations from Euclidean geometry influence our behavior. In this paper we show that the well-known law relating figural and kinematic aspects of drawing - that Euclidean tangential velocity V_e is proportional to the radius of curvature R to the 1/3 power - can be explained by looking in the affine space rather than the Euclidean one. We show that if instead of computing the Euclidean velocity we compute the affine one, a velocity which is invariant to affine transformations, then we obtain that the curve is being draw with constant velocity. Moreover, it can be shown that the unique function of R which will give an affine invariant velocity is $R^{1/3}$. This means that the velocity which was found experimentally is the unique one which tells that constant affine distances are traveled in a given time interval. Note that the planar motion of a point-light is perceived as uniform if the tangential velocity obeys the same 1/3 power law. This result can also be explained by the same proposed approach.

Why affine invariance? In vision, affine transformations are obtained when a planar object is rotated and translated in space, and then projected into the eye (camera) via a parallel projection. This is a good model of the human visual system when the object is flat enough, and away from the eye, as in the case of drawing. Accordingly, affine concepts have been applied to the analysis of image motion and the perception of three-dimensional structure from motion [14, 15, 16, 17, 18]. Another way that affine invariance could arise is that the transforms from visual input to motor output could approximate the true Euclidean transformations [19] and do so with affine approximations. Although in this work we do not attempt to isolate the stage in visuomotor processing at which the affine geometry enters, the essential explanation of the 1/3 power remains the same.

A planar curve may be regarded as the trajectory of a point $p \in [0, a]$ on the plane. For each value of p, a point $\mathcal{C}(p) = [x(p), y(p)] \in \mathbf{R}^2$ on the curve is obtained. Different parametrizations p give different velocities, but define the same trace or geometric curve. An important parametrization is the *Euclidean arc-length* v [20], which means that the curve is traveled with constant velocity. In this case the Euclidean length of a curve between v_0 and

v_1 is $l_e(v_0, v_1) = \int_{v_0}^{v_1} dv$, and the Euclidean velocity is defined via

$$V_e := \frac{dv}{dt},$$

where t stands for time. This is the classical definition of velocity, which relates the (Euclidean) distance traveled with the time it takes to travel it. This is also the velocity as interpreted in the experiments of hand writing and planar point motion, where it was found that

$$V_e = kR^{1/3}, \tag{1}$$

where k is a constant and R is the radius of curvature.

This arc-length v is Euclidean invariant. If a curve $\mathcal{C}$ is transformed to $\tilde{\mathcal{C}}$ via a rotation and a translation, that is, an Euclidean transformation, then $dv = d\tilde{v}$. From this we conclude that when distances are measured by l_e, they are Euclidean invariant.

Suppose now that instead of only rotations and translations, we have *affine* transformations, which means that the curve can be stretched with different values in the horizontal and vertical directions. For the affine group, v is not invariant any more, $dv \neq d\tilde{v}$ and $l_e \neq \tilde{l}_e$. We can define a new notion of *affine arc-length* (s), and based on it an *affine length* (l_a), which are affine invariant [21, 22]. The affine arc-length is given by the requirement $\left|\frac{\partial \mathcal{C}}{\partial s} \times \frac{\partial^2 \mathcal{C}}{\partial s^2}\right| = 1$, which means that the area of the parallelogram determined by the vectors $\frac{\partial \mathcal{C}}{\partial s}$ and $\frac{\partial^2 \mathcal{C}}{\partial s^2}$ is constant. This gives the simplest affine invariant parametrization [23]. Based on this, we define the affine invariant distance as $l_a(v_0, v_1) := \int_{v_0}^{v_1} ds$, and the affine velocity as $V_a := \frac{ds}{dt}$. The affine velocity relates the affine distance l_a with the time it takes to travel it.

Assume that the curve is parametrized via Euclidean arc-length v. Then, using the relation between an arbitrary parametrization and s [21], we have $\frac{ds}{dv} = \left|\frac{\partial \mathcal{C}}{\partial v} \times \frac{\partial^2 \mathcal{C}}{\partial v^2}\right|^{1/3} = |\vec{\mathcal{T}} \times \kappa \vec{\mathcal{N}}|^{1/3} = \kappa^{1/3}$, where $\vec{\mathcal{T}}$, $\vec{\mathcal{N}}$, and $\kappa = 1/R$ are the unit tangent, unit normal, and the Euclidean curvature respectively. Therefore

$$V_a = \frac{ds}{dt} = \frac{ds}{dv}\frac{dv}{dt} = \kappa^{1/3} V_e = \frac{1}{R^{1/3}} V_e.$$

For the case of handwriting velocity (1) we have that

$$V_a \propto k, \tag{2}$$

which means that the curve is traveled with constant affine velocity. This means for example that a circle and an ellipse will be traveled at times proportional to k, since they are related by an affine transformation.

We performed an experiment to determine if, as predicted, curves were drawn at constant affine velocity and that drawing time remained constant for shapes of equal affine length (Figure 1). Results showed that curves were drawn with constant affine velocity, but that shapes with equal affine lengths did not have equal drawing times (Figure 2). This apparent contradiction was resolved by examining errors in subjects' drawing, where it was found that cumulative error in reproducing the local shape could account for the increase in drawing times (Figure 3).

Figure Legends

Figure 1. The sixteen shapes used in the drawing experiment. Each column contains four figures with equal affine length and corresponds to an affine-transformed hippopede (See [24]; polar equation $r^2 = 4b(a - b\sin^2\theta)$). The 4 hippopedes were obtained with the values $a = 4.3$mm and $b = \frac{a}{5}, \frac{a}{4}, \frac{a}{3.25}, \frac{a}{2}$ for the columns left to right and each was rotated so that its long axis was vertically aligned. The area-preserving affine transformation used in obtaining the 4 rows was to stretch by an amount α in the vertical direction while compressing by an amount $\frac{1}{\alpha}$ in the horizontal direction. The four rows, from top to bottom, correspond to values of α=1.2, 1.85, 2.5 and 3.25. Six subjects twice traced each of the sixteen shapes as carefully as possible for a period of 45 seconds. Position data was sampled at 205 HZ from a digitizing pad with 0.02 mm accuracy and was digitally filtered with a fifth order butterworth filter with a cutoff frequency of 10 Hz. Subjects reproduced the Euclidean perimeter with an average error of 0.6 mm (SD 1.3 mm) which showed no statistically significant variation with the amount of stretch or affine length.

Figure 2. a) An example of corresponding instantaneous Euclidean and affine velocities (filtered at 1 Hz cutoff). Euclidean velocity is periodic with the drawing motion while affine velocity is roughly constant (units of velocity: Euclidean ($\frac{mm}{sec}$), affine ($\frac{mm^{2/3}}{sec}$)). b) Averages of subjects' instantaneous affine and Euclidean velocities. Average instantaneous affine velocity (open marks) was constant for all shapes while average instantaneous Euclidean velocity (filled marks) increased with the Euclidean perimeter (units of velocity: Euclidean ($\frac{mm}{sec}$), affine ($\frac{mm^{2/3}}{sec}$)). c) Average drawing time did not remain constant for shapes of equal

affine length, but increased for shapes with greater Euclidean perimeter.

Figure 3. Subjects' errors in reproducing the local form of the presented shape were related to their increase in drawing time. This can be seen by plotting the drawing times versus the average error in the total radius of curvature. This error was defined as the the sum of the radius of curvature of the drawn shape minus the approximate numerical integral of the radius of curvature of the presented shape. For each affine length, as α increased the presented curve became straighter and the amount of miss-drawing increased due to subjects inability to match this straightness. Because affine velocity remained constant, this resulted in increased affine length and thus longer drawing times.

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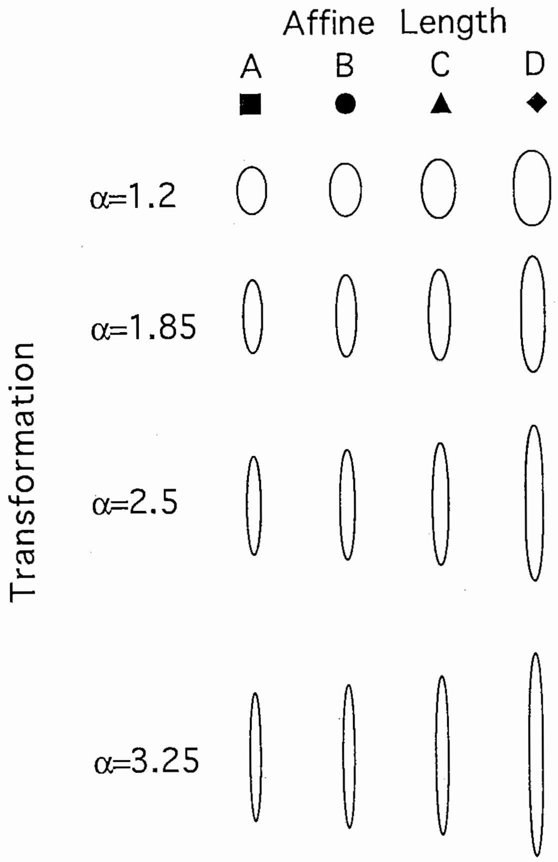

Figure 1

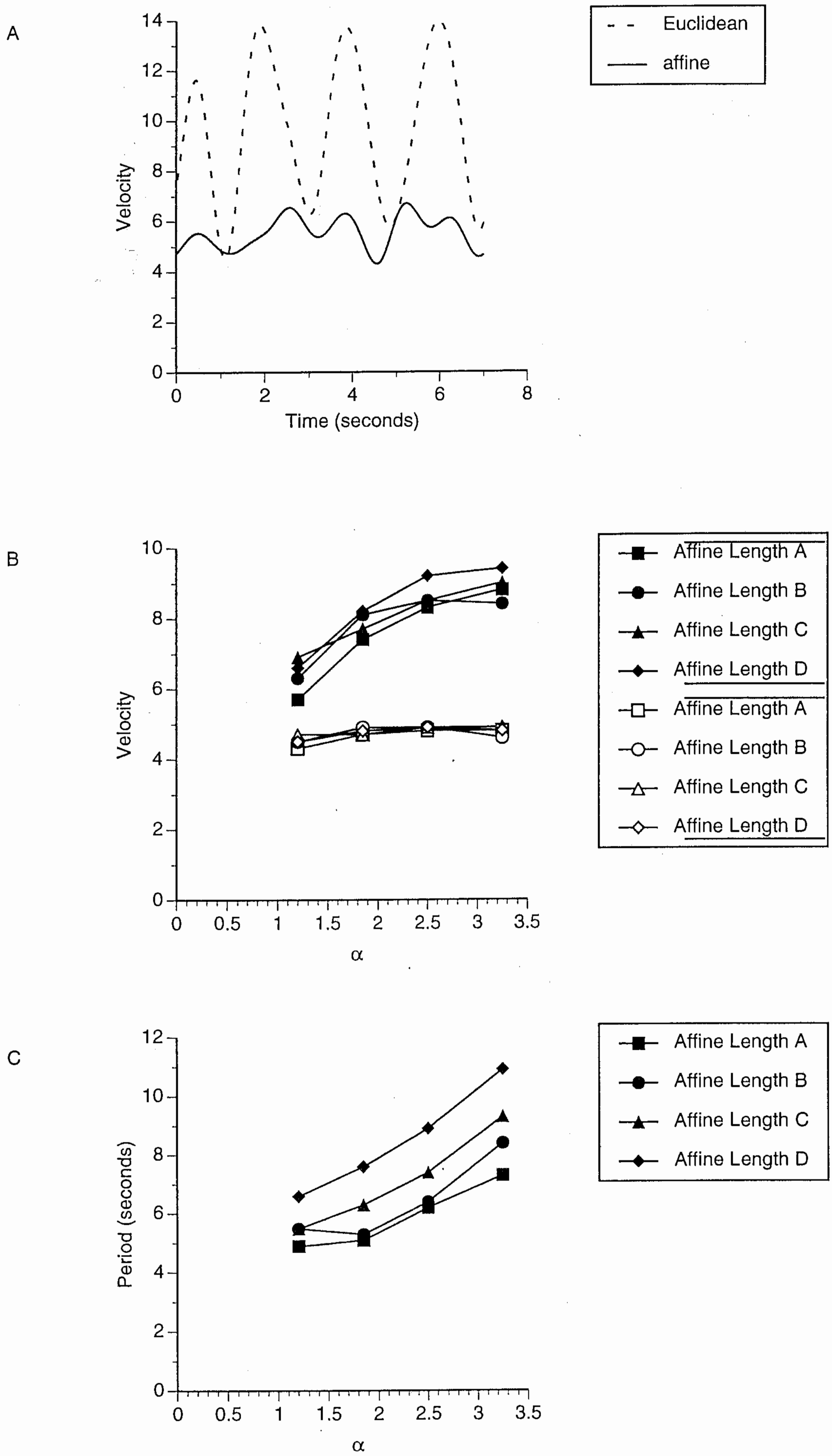
A
B
C
Velocity
Time (seconds)
Euclidean
affine
Period (seconds)
α
Affine Length A
Affine Length B
Affine Length C
Affine Length D

Figure 2

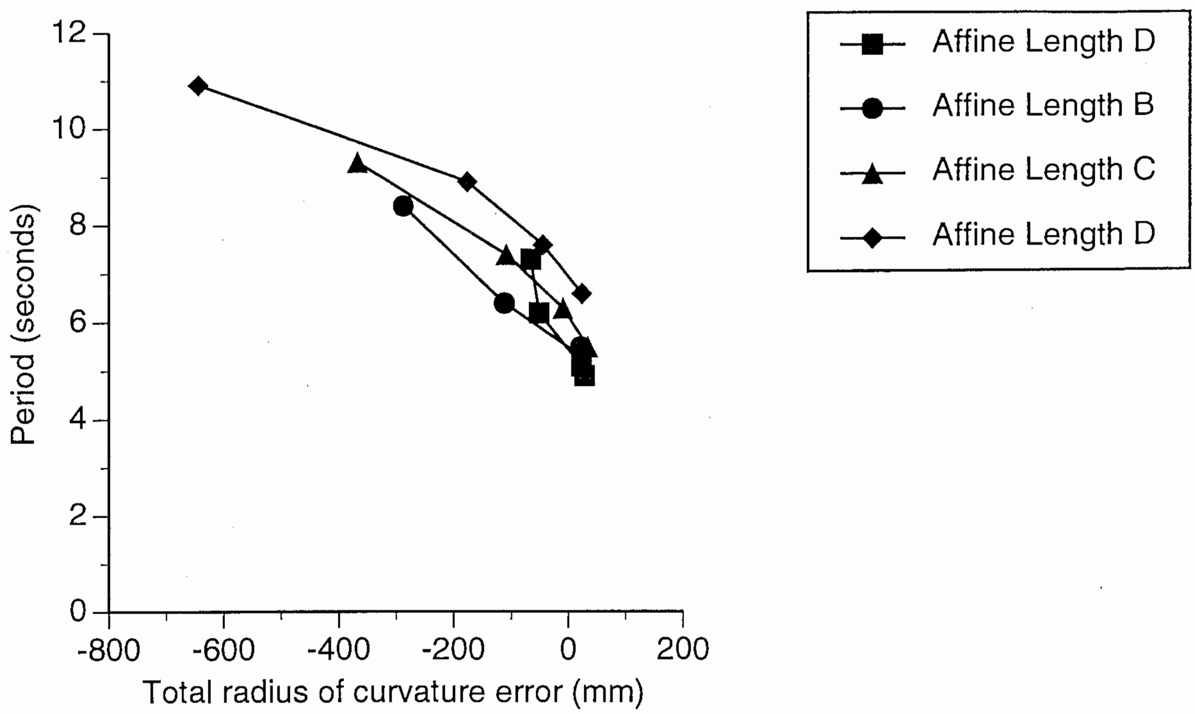
12
10
8
6
4
2
0
Period (seconds)
-800
-600
-400
-200
0
200
Total radius of curvature error (mm)
Affine Length D
Affine Length B
Affine Length C
Affine Length D

Figure 3