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The Asynchronous MFT Equation Converges Faster Than the Hopfield Network

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Abstract

The analog Hopfield network has been applied for a variety of optimization problems. Peterson and Anderson (Peterson and Anderson 1987, 1988) have shown that the Liapunov function of the Hopfield network corresponds to the free energy function in the mean field theory (MFT) of the Boltzmann machine. They proposed an asynchronous MFT equation to find local minima of the free energy function. However, convergence of the asynchronous MFT equation has not been analyzed theoretically. This article gives proof that the asynchronous MFT equation decreases the free energy by a finite amount at each time step, and converges to a local minimum of the free energy function. It is also shown that the asynchronous MFT equation converges faster than the Hopfield network.

Good solutions for large size TSP can be obtained by using a MFT for a Potts spin model. This article also provides proof that the asynchronous MFT equation for a Potts spin model converges to a local minimum of the model's free energy.

1. Introduction

In his original paper, Hopfield (Hopfield 1984) has shown that a Liapunov function can be defined for the analog Hopfield network and it always converges to a local minimum of the Liapunov function. When the slope of the sigmoid function in the Hopfield network becomes very large, the Liapunov function becomes nearly equal to the energy function, which has a quadratic form of the state variables. Therefore, the Hopfield network can be used for solving optimization problems defined as minimization of the quadratic energy function (Hopfield and Tank 1985).

The physical meaning of the Hopfield network has been further clarified by Peterson and Anderson (Peterson and Anderson 1987, 1988). They have shown that the Hopfield network is equivalent to the mean field theory (MFT) of the Boltzmann machine (Ackley et al. 1985). The Liapunov function of the Hopfield network corresponds to the free energy function in the MFT. This implies that the Hopfield network finds local minima of the free energy function in the MFT. Peterson and Anderson (Peterson and Anderson 1988) also proposed an asynchronous MFT equation which finds local minima of the free energy function. They pointed out that this asynchronous MFT equation is more efficient than the Hopfield network to find the local minima. However, convergence of the asynchronous MFT equation has not been analized theoretically. In this article, we will prove the following points:

1. The asynchronous MFT equation decreases the free energy by a finite amount at each time step.

2. The asynchronous MFT equation always converges to a local minimum of the free energy function.

3. The asynchronous MFT equation converges faster than the Hopfield network.

Peterson and Söderberg (Peterson and Söderberg 1989) have shown that good solutions for large size TSP can be obtained by using a MFT for a Potts spin model. In the appendix it is proved that the asynchronous MFT equation for a Potts spin model converges to a local minimum of the model's free energy function.

2. Mean Field Theory

In the mean field theory (MFT), the i-th unit variable V_i represents a probability that the corresponding binary variable in the Boltzmann machine takes the value 1. Accordingly, the unit variable V_i takes a value between 0 and 1. The energy E, the entropy S and the free energy F in the MFT are given by

$$F = E - TS \tag{2.1a}$$

$$E = -\frac{1}{2}\sum_{i,j=1}^{N} W_{ij}V_iV_j - \sum_{i=1}^{N} I_iV_i \tag{2.1b}$$

$$S = -\sum_{i=1}^{N}\left(V_i \log V_i + (1-V_i)\log(1-V_i)\right) \tag{2.1c}$$

, where T represents temperature, and there is no self-connection:

$$W_{ii} = 0 \qquad (i = 1,\cdots,N). \tag{2.1d}$$

The local minimum of the free energy F is given by

$$\frac{\partial F}{\partial V_i} = -\sum_{j=1}^{N} W_{ij}V_j - I_i + T\log(V_i/(1-V_i)) = 0 \tag{2.2}$$

for all i. By introducing auxiliary variables U_i, (2.2) can be rewritten as

$$U_i = \sum_{j\neq i} W_{ij}V_j + I_i \tag{2.3a}$$

$$V_i = G(U_i) = 1/\left(1+\exp(-U_i/T)\right) \qquad (i = 1,\cdots,N). \tag{2.3b}$$

This equation is called the MFT equation (Peterson and Anderson 1987, 1988). From (2.3b), one can see that the inverse temperature (1/T) represents the slope parameter of the sigmoid function G. The solution of (2.3) can be obtained by using the Hopfield network (Hopfield 1984):

$$\tau\frac{dU_i(t)}{dt} = -U_i(t) + \sum_{j\neq i} W_{ij}V_j(t) + I_i \tag{2.4a}$$

$$= -\partial F/\partial V_i \qquad (i = 1,\cdots,N). \tag{2.4b}$$

By using (2.4b),

$$\frac{dF}{dt} = \sum_{i=1}^{N} \frac{\partial F}{\partial V_i} \frac{dV_i}{dt} = -\frac{1}{\tau} \sum_{i=1}^{N} \left(\frac{\partial F}{\partial V_i} \right)^2 G'(U_i) \leq 0 \tag{2.5}$$

is derived, where G' represents the derivative of the sigmoid function G. This implies that the Hopfield network always decreases the free energy F and converges to a local minimum of the free energy F (Hopfield 1984). Therefore, the Liapunov function of the Hopfield network is given by the free energy F in the MFT.

The solution of the MFT equation (2.3) can be also obtained by the following asynchronous iterative equation (Peterson and Anderson 1987, 1988).

$$V_i(t) = G\left(\sum_{j \neq i} W_{ij} V_j(t - \Delta t) + I_i \right) \tag{2.6a}$$

$$V_j(t) = V_j(t - \Delta t) \quad \text{for} \quad j \neq i. \tag{2.6b}$$

In this equation, only one variable V_i is updated at time t, and the other variables $V_j, j \neq i$, are not changed at this point. All the variables are updated once in N time steps. The equation (2.6) is called the asynchronous MFT equation.

3. Convergence of the asynchronous MFT equation

A local minimum of the free energy F is a fixed point of the asynchronous MFT equation (2.6). However, this fixed point may be unstable and a limit cycle solution may appear for a general iterative nonlinear map such as (2.6). In the following, we will prove that the asynchronous MFT equation always converges to a local minimum of the free energy F. From (2.6),

$$\frac{\partial F}{\partial V_i}(V(t)) = 0 \tag{3.1}$$

holds since V_i does not appear in the right hand side of (2.6a). Since

$$\frac{\partial^2 F}{\partial V_i^2} = \frac{T}{V_i(1 - V_i)} > 0, \tag{3.2}$$

the free energy function F is convex along the V_i coordinate and has a unique minimum along this coordinate, when the other variables $V_j, j \neq i$, are all fixed. Therefore, (3.1) implies that the free energy is minimized with respect to the V_i coordinate at time t. In other words, the free energy F either decreases by a finite amount at each time step or is not changed when $\partial F / \partial V_i = 0$ for the current V_i variable. The amount of decrease is maximal under the asynchronous update. If the current state is not a local minimum of the free energy F, the state must change in N time steps, since $\partial F / \partial V_i \neq 0$ for some V_i. On the other hand, if the current state is a local minimum of the free energy F, the state

never changes. Consequently, the asynchronous MFT equation (2.6) always converges to a local minimum of the free energy F.

In numerical simulation, the Hopfield network (2.4) must be translated into a discrete time equation. By using the Euler method, an asynchronous Hopfield equation is given by

$$U_i(t) = (1 - \Delta t/\tau)U_i(t - \Delta t) + (\Delta t/\tau)\left(\sum_{j \neq i} W_{ij}V_j(t - \Delta t) + I_i\right), \tag{3.3a}$$

$$V_i(t) = G(U_i(t)), \tag{3.3b}$$

$$V_j(t) = V_j(t - \Delta t) \qquad (j \neq i). \tag{3.3c}$$

If the time step Δt is sufficiently small, the free energy decreases for each time step. However, the amount of decrease is also infinitesimal. In the asynchronous MFT equation, the amount of decrease is maximal under the constraint that only the V_i variable is changed. Therefore, the asynchronous MFT equation converges faster than the asynchronous Hopfield network. For the synchronous Hopfield network, the situation is the same. In order to guarantee that the free energy decreases, the time step Δt must be sufficiently small, so the amount of decrease is also infinitesimal. Therefore, the synchronous Hopfield network is also inefficient compared to the asynchronous MFT equation.

The synchronous MFT equation has some problems. Since all the variables are updated at once according to (2.6a) in the synchronous mode, equation (3.1) no longer holds and decrease in the free energy is not guaranteed for the synchronous MFT equation. In general, the synchronous MFT equation may exhibit limit cycle solutions.

4. Conclusion

The advantage of the asynchronous MFT equation is that it decreases the free energy by the maximal amount under the asynchronous update. Therefore, the asynchronous MFT equation converges faster than the Hopfield network, which decreases the free energy by an infinitesimal amount.

Appendix

The MFT free energy for a Potts spin model (Peterson and Söderberg 1989) is given by

$$F = -\frac{1}{2}\sum_{i,j,a,b} W_{ia,jb}V_{ia}V_{jb} - \sum_{i,a} V_{ia}I_{ia} + T\sum_{i,a} V_{ia}\log V_{ia} \tag{A.1}$$

where i, j runs 1 to N and a, b runs 1 to M. The Potts spin variable V_{ia} in the MFT satisfies the following constraint

$$\sum_{a=1}^{M} V_{ia} = 1 \qquad (i = 1, \cdots, N) \tag{A.2}$$

and takes a value between 0 and 1. In order to find a local minimum of the free energy (A.1) under the constraint (A.2), it is convenient to solve the constraint equation (A.2) explicitly.

$$V_{iM} = 1 - \sum_{a<M} V_{ia} \ , \tag{A.3}$$

$$\frac{\partial}{\partial V_{ia}}\Big|_c = \frac{\partial}{\partial V_{ia}} - \frac{\partial}{\partial V_{iM}} \ , \qquad (a < M) \tag{A.4}$$

where $|_c$ represents a derivative under the constraint (A.2). A local minimum of the free energy (A.1) satisfies the following stationary condition.

$$\frac{\partial F}{\partial V_{ia}}\Big|_c = \frac{\partial F}{\partial V_{ia}} - \frac{\partial F}{\partial V_{iM}} = 0. \qquad (a < M) \tag{A.5}$$

By defining Q_i as $Q_i = \partial F / \partial V_{iM}$, the stationary condition (A.5) can be written as

$$Q_i = \frac{\partial F}{\partial V_{ia}} = -\sum_{j,b} W_{ia,\,jb} V_{jb} - I_{ia} + T(\log V_{ia} + 1). \tag{A.6}$$

The value of the auxiliary variable Q_i, which corresponds to the Lagrange multiplier, is determined by the constraint (A.2). Consequently, (A.6) is rewritten as

$$U_{ia} = \sum_{j,b} W_{ia,\,jb} V_{jb} + I_{ia} \ . \tag{A.7a}$$

$$V_{ia} = H(U_i) = e^{U_{ia}/T} / (\sum_{b} e^{U_{ib}/T}). \tag{A.7b}$$

This is the MFT equation for the Potts spin model (A.1). An asynchronous MFT equation for (A.7) is given by

$$U_{ia}(t) = \sum_{j,b} W_{ia,\,jb} V_{jb}(t - \Delta t) + I_{ia}, \tag{A.8a}$$

$$V_{ia}(t) = H(U_i(t)), \tag{A.8b}$$

$$V_{jb}(t) = V_{jb}(t - \Delta t) \qquad \text{for } j \neq i. \tag{A.8c}$$

If the connection matrix W satisfies the condition

$$W_{ia,\,ib} = 0 \qquad \text{for any } i, a, b, \tag{A.9}$$

the equation

$$\frac{\partial F}{\partial V_{ia}}\Big|_c(V(t)) = 0 \qquad \text{for } a < M \tag{A.10}$$

holds. The curvature of the free energy in the subspace, V_{ia} (i: fixed), is given by

$$\frac{\partial^2 F}{\partial V_{ia}{}^2}\Big|_c = T\left(\frac{1}{V_{ia}} + \frac{1}{V_{iM}}\right) \quad , \tag{A.11a}$$

$$\frac{\partial^2 F}{\partial V_{ia}\partial V_{ib}}\Big|_c = T / V_{iM} \qquad (a \neq b). \tag{A.11b}$$

From this equation, it is shown that the second variation of F is always positive:

$$\delta^2 F = \sum_{a,b<M} \frac{\partial^2 F}{\partial V_{ia}\partial V_{ib}}\Big|_c \delta V_{ia}\delta V_{ib}$$

$$= \sum_{a<M} \frac{T}{V_{ia}} \delta V_{ia}{}^2 + \frac{T}{V_{iM}} \left(\sum_{a<M} \delta V_{ia}\right)^2 > 0. \tag{A.12}$$

Therefore, the free energy F (A.1) is convex and has only one minimum when V_{jb} ($j \neq i$) is fixed. From this fact and (A.10), it is shown that the asynchronous MFT equation (A.8) minimizes the free energy F in the subspace V_{ia} (i: fixed). By using the same argument as in the previous section, it is shown that the asynchronous MFT equation (A.8) converges to a local minimum of the free energy F (A.1).

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