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Abstract

We often treat information that was projected into a subspace from a space where the original information exists. For example, visual information is information that was projected onto the retina from the 3-dimensional Euclidean space. Because there is a deficiency of information caused by the projection, we can not uniquely recover the original information in general. Therefore, it is definitely important to find properties, if any, that essentially connect the original information with the projected information. When a class of admissible transformations to which the original information is subject is specified, projective invariants, which are real-valued functions in terms of the projected information and which are unaffected by the class of admissible transformations, provide an essential relationship between the original information and the projected one. This paper is a study on projective invariants under the condition that the n-dimensional projective space is projected into the $(n-1)$-dimensional projective space by the projection of a certain class; and that the class of admissible transformations involves projective transformations in the n-dimensional projective space. It is shown that, for given integers i and j such that $1 \leq i \leq j \leq n-i$, we have a projective invariant derived from $(n+i+j)$ subspaces of $(n-2)$ dimensions, where the $(n+i+j)$ subspaces are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes arranged in the letter H. The nonsingularity condition, i.e., the condition under which the invariant is nonsingular, is also given.

Key Words: projective invariants, admissible transformations, interpretation vector, intersections of hyperplanes, nonsingularity condition.

Contents

1 Introduction

When we observe a subject under investigation, we often only obtain a certain part of the original information, i.e., information projected into a subspace from a space where the original information exists. We are then required to deal with such partial information to investigate the subject. For instance, in observing objects in three dimensions, we obtain visual information that was projected onto the retina from the 3-dimensional Euclidean space; we have to recognize the objects by making use of the projected information on the retina. Because there is a deficiency of information caused by the projection, the problem of recovering the original information is ill-posed: in general, we can not uniquely recover the original information from the projected information. In addition, when a transformation operates on the original information, the projected information before and after the transformation differs. In other words, the projected information significantly varies, depending on the transformation that operates on the original information, even for the same original information. Thus, it is important to find properties, if they exist, that essentially connect the original information with the projected information.

When original information in a space is subject to a given class of admissible transformations, *projective invariants*, which are real-valued functions in terms of the projected information and which are unaffected by the class of admissible transformations, provide an essential relationship between the original information and the projected one. When we can not directly deal with the original information, such projective invariants play an important role in investigating the properties of the original information. For example, for object recognition problem which is one of the most important problems in computer vision, projective invariants aid in identifying one object out of many (hence, the importance of projective invariants has been continually emphasized since the origin of the field of computer vision in the 1960s) [2], [5], [6], [7].

On the other hand, invariants were a very active mathematical subject in the latter half of the 19th century [4]. However, they were not derived through projections: they were derived not by dealing with the projected information but with the original information itself. Therefore, invariants [1], [3] that were studied then are nothing but invariants of admissible transformations themselves. In contrast to this, in practice, we often face situations in which we have to get at the essence of the original information by way of the projected information, and we can not deal with the original information. Hence, investigating the existence of projective invariants is very significant from the engineering point of view.

In this paper we consider the existence of projective invariants under the condition that $(n-2)$-dimensional subspaces in the $(n-1)$-dimensional projective space were projected from the n-dimensional projective space by the projection of a certain class; and that the inverse images of these subspaces with respect to the projection are subject to projective transformations in the n-dimensional projective space. We are mainly interested in deriving projective invariants in a concrete fashion in terms of $(n-2)$-dimensional subspaces in the $(n-1)$-dimensional projective space.

The main theorems, which are given in §3, state that (1) for given integers i and j such that $1 \leq i \leq j \leq n-i$, we have a projective invariant derived from $(n+i+j)$ subspaces of $(n-2)$ dimensions, where the $(n+i+j)$ subspaces are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes arranged in the letter H; and (2) the projective invariant is nonsingular, i.e., well-defined and nondegenerated, iff (COND) below is satisfied by n subspaces among the $(n+i+j)$ subspaces, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part of H, in the arrangement (we always have four cases).

(COND) Not singular is an $(n+1) \times (n+1)$ matrix whose column vectors are the homogeneous coordinates of $(n+1)$ hyperplanes that determine n subspaces of $(n-2)$ dimensions.

In this paper, when an arrangement of hyperplanes or $(n-2)$-dimensional subspaces has the same topology as the letter H, we call "they are arranged in the letter H"; hence, they could n-dimensionally exist. (1) indicates that we have a projective invariant of $(n+i+j)$ subspaces of $(n-2)$ dimensions arranged in the letter H (accordingly, the $(n+i+j)$ subspaces could n-dimensionally exist). It should be noted that the number of $(n-2)$-dimensional subspaces in the left part of H is $2i$, whereas that in the right part is $2j$; and, furthermore, the arrangement is symmetrical with respect to the horizontal part of H. In addition, the number of this kind of invariants in the n-dimensional projective space is $\lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor\right)$ (see Page 7 for the notation). (2) implies that our invariant is almost always nonsingular when we randomly choose $(n+i+j+1)$ hyperplanes in the n-dimensional projective space. This is because the homogeneous coordinates of $(n+1)$ hyperplanes randomly chosen in the n-dimensional projective space are linearly independent in general.

This paper is organized as follows. In §2, we formulate the problem to solve. In §3, the results of this paper, i.e., the existence of projective invariants and the nonsingularity condition for our invariants are presented as two theorems. Their proofs are given in §4.

2 Problem Formulation

Let $\mathbf{P}^n$ be the n-dimensional projective space over the real number field $\mathbf{R}$. We assume $n \geq 3$ throughout the paper. Note that if not explicitly stated, the coordinates of a point are understood to be homogeneous.

Letting $\boldsymbol{c} = (1, 0, 0, \ldots, 0)^{\mathrm{T}}$ $(\in \mathbf{P}^n)$, we consider the set of mappings : $\mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$ as follows.

$$\mathcal{F} \ := \ \{f_P \mid P \in \mathrm{PGL}(n-1)\},$$

where $\mathrm{PGL}(n-1)$ denotes the projective general linear group of degree $(n-1)$ over $\mathbf{R}$; and f_P is a mapping : $\mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$ that is represented by $n \times (n+1)$ matrix F_P:

$$F_P \ = \ \left(0 \middle| \ \ P \ \ \right) \quad (P \in \mathrm{PGL}(n-1)).$$

Therefore, when we put $\boldsymbol{x} \in \mathbf{P}^n - \{\boldsymbol{c}\}$ and $\boldsymbol{X} = f_P(\boldsymbol{x})$, then we have

$$\rho \boldsymbol{X} \ = \ F_P \boldsymbol{x} \qquad (\rho \in \mathbf{R}^*),$$

where $\mathbf{R}^*$ denotes the set of nonzero real numbers. In this paper, we are interested in the class $\mathcal{F}$ of mappings : $\mathbf{P}^n - \{\boldsymbol{c}\} \longrightarrow \mathbf{P}^{n-1}$; and we call an element of $\mathcal{F}$ a *projection*. We assume that we can deal only with $\boldsymbol{X}$, i.e., the image of $\boldsymbol{x}$ projected by f_P where f_P is derived from a given $P \in \mathrm{PGL}(n-1)$ as seen above. It should be noted that, when we denote by I the unit matrix of degree n, $\forall F_P$ is expressed by

$$F_P \ = \ P\, F_I.$$

If we restrict $\mathbf{P}^n - \{\boldsymbol{c}\}$ and $\mathbf{P}^{n-1}$ to the n-dimensional vector space over R that excludes the origin and hyperplane $x_1 = 0$; and to the $(n-1)$-dimensional vector space over R, respectively,

f_I ($\in \mathcal{F}$) coincides with the central projection where the center of the projection is the origin (its coordinates in $\mathbf{P}^n$ are $\boldsymbol{c}$), and where the projection hyperplane is $x_1 = 1$ (see Fig. 1).

Let $\mathcal{T}$ be the set of projective transformations for $\mathbf{P}^n - \{\boldsymbol{c}\}$:

$$\mathcal{T} \;=\; \{T \mid T \,:\, \mathbf{P}^n - \{\boldsymbol{c}\} \to \mathbf{P}^n,\; T \in \mathrm{PGL}(n)\,\}.$$

For $S \subseteq \mathbf{P}^n - \{\boldsymbol{c}\}$, we define

$$\mathcal{T}_S \;:=\; \{T \mid T \in \mathcal{T};\; T(\boldsymbol{x}) \neq \boldsymbol{c},\; \forall \boldsymbol{x} \in S\}.$$

Since $\mathcal{T}_S$ forms a group, we set $\mathcal{T}_S$ to be the class of admissible transformations for S. In addition, we put

$$f_P(S) \;:=\; \bigcup_{\boldsymbol{x} \in S} \{f_P(\boldsymbol{x})\}.$$

In accordance with the notations introduced above, we formulate our problem, namely, the problem of finding a real-valued function which is defined in terms of the images of S projected by f_P; and which is unaffected by $\mathcal{T}_S$, i.e., the class of admissible transformations.

Problem 2.1 Let $f_P \in \mathcal{F}$ and S ($\subseteq \mathbf{P}^n - \{\boldsymbol{c}\}$) be given. Find a natural number N and a function $Inv : \overbrace{f_P(S) \times f_P(S) \times \cdots \times f_P(S)}^{N} \longrightarrow \mathbf{R}$ such that,

for $\forall T \in \mathcal{T}_S$,

$$Inv\,(f_P(\boldsymbol{x}),\, f_P(\boldsymbol{x}), \ldots, f_P(\boldsymbol{x})) \;=\; Inv\,(f_P(T(\boldsymbol{x})),\, f_P(T(\boldsymbol{x})), \ldots, f_P(T(\boldsymbol{x})))\,,$$

where $\boldsymbol{x} \in S$. □

Function Inv is a projective invariant under the condition that the projection is achieved by f_P, and the class of admissible transformations is $\mathcal{T}_S$ for a given S. Our aim in this paper is, for given f_P and S, to find natural number N and function Inv in Problem 2.1. For $\forall f_P \in \mathcal{F}$, $(n-2)$-dimensional subspaces in $\mathbf{P}^n - \{\boldsymbol{c}\}$ are projected into $(n-2)$-dimensional subspaces in $\mathbf{P}^{n-1}$ by f_P; and we can deal with the projected $(n-2)$-dimensional subspaces[1]. Hence, we set f_P to be f_{P_*}, that is, f_{P_*} derived from an arbitrary $P_* \in \mathrm{PGL}(n-1)$; and S to be

[1]Let $f_{P_1}, f_{P_2} \in \mathcal{F}$, then an image of a point in $\mathbf{P}^n - \{c\}$ projected by f_{P_1} is connected to that projected by another projection f_{P_2} through a projective transformation in $\mathbf{P}^{n-1}$ (an element of $\mathrm{PGL}(n-1)$; to be more specific, $P_1 P_2^{-1}$ or $P_2 P_1^{-1}$).

the set whose elements are N subspaces, which n-dimensionally exist, of $(n-2)$ dimensions in $\mathbf{P}^n - \{c\}$. We then focus on finding a real-valued function having the following properties: 1) it is defined in terms of the coefficients of the equations that determine the N projected subspaces of $(n-2)$ dimensions, and 2) its value remains invariant even if the inverse images with respect to $f_{P_\bullet}$ are transformed by any admissible transformation, i.e., any element of $\mathcal{T}_S$.

3 Results

The results of this paper are presented as Theorems 3.1 and 3.2. Their proofs are postponed until the next section.

For an $(n-2)$-dimensional subspace

$$\sum_{\kappa=0}^{n-1} a_\kappa X_\kappa = 0 \tag{3.1}$$

in $\mathbf{P}^{n-1}$ (its coordinate system is $(X_0, X_1, \ldots, X_{n-1})^{\mathrm{T}}$), where

$$\sum_{\kappa=0}^{n-1} {a_\kappa}^2 \neq 0,$$

we obtain a vector $(a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$ that is determined by the coefficients of the equation. We call this vector the *interpretation vector* of the subspace. The interpretation vector is the homogeneous coordinates of the subspace.

Remark 3.1 We can only determine vector $(a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$ up to a scaling factor when we actually observe subspace (3.1) in $\mathbf{P}^{n-1}$. However, we can eliminate this indeterminacy by setting a criterion such as $a_0 = 1$ or the normalization of the vector. □

An $(n-2)$-dimensional subspace in S is uniquely determined as the intersection of a pair of hyperplanes in $\mathbf{P}^n - \{c\}$ (see Fig. 2). Thus, we represent an element of S as a pair of hyperplanes in $\mathbf{P}^n - \{c\}$. Let $\boldsymbol{n}_{ij}$ denote the interpretation vector of the intersection subspace of two hyperplanes i and j in $\mathbf{P}^n - \{c\}$.

For two integers i and j such that $1 \leq i \leq j \leq n-i$, we define the following sets of hyperplanes in $\mathbf{P}^n - \{c\}$.

$$\Omega_{\mathrm{L1}} := \{\mathrm{L1}_1, \mathrm{L1}_2, \ldots, \mathrm{L1}_i\},$$

$$\Omega_{\mathrm{L2}} := \{\mathrm{L2}_1, \mathrm{L2}_2, \ldots, \mathrm{L2}_i\},$$

$$\Omega_{R1} := \{R1_j, R1_{j-1}, \ldots, R1_1\},$$
$$\Omega_{R2} := \{R2_j, R2_{j-1}, \ldots, R2_1\},$$
$$\Omega_{C} := \{C_1, C_2, \ldots, C_{n+1-i-j}\},$$

where $L1_\lambda, L2_\lambda, R1_\mu, R2_\mu, C_\nu$ ($\lambda \in \{1,2,\ldots,i\}; \mu \in \{1,2,\ldots,j\}; \nu \in \{1,2,\ldots,n+1-i-j\}$) are all natural numbers; and any two of Ω_ℓ ($\ell \in \{L1, L2, R1, R2, C\}$) are disjoint. Note that $|\Omega_{Lk}|+|\Omega_C|+|\Omega_{R\ell}| = n+1$ ($k,\ell \in \{1,2\}$). It is important to remark that we assume that the order of elements of Ω_ℓ ($\ell \in \{L1, L2, R1, R2, C\}$) makes sense. Namely, hyperplanes in Ω_ℓ are assumed to be aligned with the order of the elements with which Ω_ℓ is defined. This should be applied to the union of Ω_ℓ's such as $\Omega_{L1} \cup \Omega_C$. Here, we suppose that $(n+1)$ different hyperplanes $\Omega_{Lk} \cup \Omega_C \cup \Omega_{R\ell}$ in $\mathbf{P}^n - \{c\}$ are given where $k,\ell \in \{1,2\}$; and n subspaces of $(n-2)$ dimensions are observed in $\mathbf{P}^{n-1}$, all of which are the images of the intersections of the adjacent hyperplanes in $\Omega_{Lk} \cup \Omega_C \cup \Omega_{R\ell}$ projected by f_{P_*}. We then consider the interpretation vectors, $\boldsymbol{n}_{Lk_1Lk_2}, \ldots, \boldsymbol{n}_{Lk_{i-1}Lk_i}, \boldsymbol{n}_{Lk_iC_1}, \boldsymbol{n}_{C_1C_2}, \ldots, \boldsymbol{n}_{C_{n-i-j}C_{n+1-i-j}}, \boldsymbol{n}_{C_{n+1-i-j}R\ell_j}, \boldsymbol{n}_{R\ell_jR\ell_{j-1}}, \ldots, \boldsymbol{n}_{R\ell_2R\ell_1}$, of the n intersection subspaces; and define an $n \times n$ matrix $N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ whose column vectors are these n vectors:

$$N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} := \Big[\boldsymbol{n}_{Lk_1Lk_2} \,|\cdots|\, \boldsymbol{n}_{Lk_{i-1}Lk_i} \,|\, \boldsymbol{n}_{Lk_iC_1} \,|\, \boldsymbol{n}_{C_1C_2} \,|\cdots|\, \boldsymbol{n}_{C_{n-i-j}C_{n+1-i-j}} \,|\, \boldsymbol{n}_{C_{n+1-i-j}R\ell_j} \,|\, \boldsymbol{n}_{R\ell_jR\ell_{j-1}} \,|\cdots|\, \boldsymbol{n}_{R\ell_2R\ell_1} \Big].$$

We attach $'$ (prime) to the notations above in the case where an admissible transformation has operated on S.

Theorem 3.1 For two integers i and j such that $1 \le i \le j \le n-i$, let $\Omega_{R1}, \Omega_{R2}, \Omega_C, \Omega_{L1}, \Omega_{L2}$ above be given sets of hyperplanes in $\mathbf{P}^n - \{c\}$; and let these sets be arranged in the letter H (see Fig. 3). Suppose that $\mathrm{rank} N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = n$ $(k,\ell \in \{1,2\})$. Then, for $(n+i+j)$ subspaces of $(n-2)$ dimensions that are the intersections of the adjacent hyperplanes in the arrangement, we have, independent of f_{P_*},

$$\begin{aligned} \mathrm{rank} N'_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} &= n, \\ \frac{\det N_{\Omega_{L1},\Omega_C,\Omega_{R1}} \cdot \det N_{\Omega_{L2},\Omega_C,\Omega_{R2}}}{\det N_{\Omega_{L1},\Omega_C,\Omega_{R2}} \cdot \det N_{\Omega_{L2},\Omega_C,\Omega_{R1}}} &= \frac{\det N'_{\Omega_{L1},\Omega_C,\Omega_{R1}} \cdot \det N'_{\Omega_{L2},\Omega_C,\Omega_{R2}}}{\det N'_{\Omega_{L1},\Omega_C,\Omega_{R2}} \cdot \det N'_{\Omega_{L2},\Omega_C,\Omega_{R1}}}. \end{aligned} \tag{3.2}$$

□

Theorem 3.1 shows that for any element of $\mathcal{F}$ (which is a projection from $\mathbf{P}^n - \{c\}$ to $\mathbf{P}^{n-1}$) there exists a projective invariant, independent of the element,

$$(3.3) \qquad Inv_{ij} := \frac{\det N_{\Omega_{L1},\Omega_C,\Omega_{R1}} \cdot \det N_{\Omega_{L2},\Omega_C,\Omega_{R2}}}{\det N_{\Omega_{L1},\Omega_C,\Omega_{R2}} \cdot \det N_{\Omega_{L2},\Omega_C,\Omega_{R1}}} \quad (1 \leq i \leq j \leq n-i)$$

for $(n+i+j)$ subspaces of $(n-2)$ dimensions, all of which are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes (in $\mathbf{P}^n - \{c\}$) arranged in the letter H (see Fig. 3). It is important to remark that we accordingly have $(n+i+j)$ subspaces of $(n-2)$ dimensions arranged in the letter H (hence, the $(n+i+j)$ subspaces could n-dimensionally exist); and also remark that the number of subspaces in the left-upper part of H is equal to that in the left-lower part: i. Whereas, the number of subspaces in the right-upper part of H is equal to that in the right-lower part: j. Namely, the arrangement is symmetrical with respect to the horizontal part of H. Therefore, for $\forall f_P \in \mathcal{F}$, when we set S to be the set whose elements are N subspaces of $(n-2)$ dimensions in $\mathbf{P}^n - \{c\}$ arranged in the letter H, N and Inv in Problem 2.1 are respectively given by $N = n+i+j$ and (3.3), where i and j are given integers such that $1 \leq i \leq j \leq n-i$. We should note that $n+2 \leq N \leq 2n$.

Remark 3.2 Since $i+j=n$ is possible, we could have $|\Omega_C| = 1$. Namely, for the $(n-2)$-dimensional subspaces arranged in the letter H, the part that corresponds to the horizontal part of H could be empty. □

For each i, we have $\xi_i = (n-2i+1)$ invariants. Taking symmetry into consideration, i can be any of $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Hence, the number[2] Ξ of this kind of invariants in $\mathbf{P}^n - \{c\}$ is given by

$$\begin{aligned} \Xi &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \xi_i \\ &= \lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor \right), \end{aligned}$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the maximum integer which is not greater than $\frac{n}{2}$.

Furthermore, we give the nonsingularity condition for Inv_{ij} $(1 \leq i \leq j \leq n-i)$, i.e., the necessary and sufficient condition under which the invariant Inv_{ij} is nonsingular. Here,

[2] In particular, we have $\lfloor \frac{n}{2} \rfloor$ invariants for $2n$ subspaces of $(n-2)$ dimensions; whereas we have only one invariant for $(n+2)$ subspaces of $(n-2)$ dimensions.

we define "an invariant is nonsingular" as "the value of the invariant is not 0, ∞ or $^0/_0$". Nonsingularity can be regarded as nondegeneracy and well-definedness. As we can see, the nonsingularity condition for an invariant ensures that the values of the invariant are numerically stable when they are calculated in practical situations. The next theorem indicates that the nonsingularity condition for invariant Inv_{ij} is almost always satisfied, when we randomly choose $(n+i+j+1)$ hyperplanes in $\mathbf{P}^n - \{\boldsymbol{c}\}$. This is because the homogeneous coordinates of $(n+1)$ hyperplanes that were randomly chosen in $\mathbf{P}^n - \{\boldsymbol{c}\}$, are linearly independent in general. Note that $(n+i+j+1)$ hyperplanes arranged in the letter H could n-dimensionally exist.

Theorem 3.2 **[Nonsingularity condition]**

Let $(n+i+j+1)$ hyperplanes where $(n+i+j)$ subspaces of $(n-2)$ dimensions exist, be arranged in the letter H (see Fig. 3). Inv_{ij} in (3.3) is nonsingular iff (COND) below is satisfied by n subspaces among the $(n+i+j)$ subspaces, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part Ω_{C} of H, in the arrangement (we always have four cases).

(COND) Not singular is an $(n+1)\times(n+1)$ matrix whose column vectors are the homogeneous coordinates (in $\mathbf{P}^n - \{\boldsymbol{c}\}$) of $(n+1)$ hyperplanes that determine n subspaces of $(n-2)$ dimensions. □

4 Proofs

The proofs for Theorems 3.1 and 3.2 are given.

First, we consider the meaning of the interpretation vector of an $(n-2)$-dimensional subspace in $\mathbf{P}^{n-1}$. Let $\boldsymbol{X}$ ($\boldsymbol{X} \neq 0$) be the coordinates (in $\mathbf{P}^{n-1}$) of any point in the subspace, and put $\tilde{\boldsymbol{X}} = P_*^{-1}\boldsymbol{X} \left(= (\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1})^{\mathrm{T}}\right)$. Then, $(1, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1})^{\mathrm{T}}$ is the inverse image of $\boldsymbol{X}$ with respect to f_{P_*}. In other words, a point in $\mathbf{P}^n - \{\boldsymbol{c}\}$ whose coordinates are $(1, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1})^{\mathrm{T}}$ is projected to a point in the subspace (in $\mathbf{P}^{n-1}$) by f_{P_*}. Moreover, put $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$ and $\tilde{\boldsymbol{a}} = P_*^{\mathrm{T}}\boldsymbol{a} \left(= (\tilde{a}_0, \tilde{a}_1 \ldots, \tilde{a}_{n-1})^{\mathrm{T}}\right)$, then (3.1) is rewritten as

$$(4.1) \qquad (0, \tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1})^{\mathrm{T}} \cdot (1, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1})^{\mathrm{T}} = 0.$$

(4.1) represents the hyperplane in $\mathbf{P}^n$ on which both $\boldsymbol{c}$ and the subspace (3.1) are (see Fig. 4). This hyperplane is called the *interpretation hyperplane* of subspace (3.1). $(0, \tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1})^{\mathrm{T}}$

is the homogeneous coordinates (or equivalently the normal vector) in $\mathbf{P}^n$ of the interpretation hyperplane of subspace (3.1). From[3] $\boldsymbol{a} = P_*^{-\mathrm{T}}\tilde{\boldsymbol{a}} = P_*^{-\mathrm{T}}F_I\,(0, \tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1})^{\mathrm{T}}$, we can see that $\boldsymbol{a}$ is obtained by applying operation $f_{P_*^{-\mathrm{T}}}$ to the homogeneous coordinates of the interpretation hyperplane of subspace (3.1). Hence, the interpretation vector of a subspace is the vector that is obtained as a result of applying operation $f_{P_*^{-\mathrm{T}}}$ to the homogeneous coordinates (in $\mathbf{P}^n$) of the interpretation hyperplane of the subspace.

As seen above, we have represented an $(n-2)$-dimensional subspace in S as a pair of hyperplanes in $\mathbf{P}^n - \{\boldsymbol{c}\}$. Thus, we next consider the relationship between the interpretation vector of the intersection subspace of two hyperplanes; and the homogeneous coordinates (in $\mathbf{P}^n - \{\boldsymbol{c}\}$) of the two hyperplanes. Let hyperplane ℓ ($\ell = \mu, \nu$) in $\mathbf{P}^n - \{\boldsymbol{c}\}$ be the set of points with coordinates $\boldsymbol{x}$ satisfying

$$\boldsymbol{a}_\ell \cdot \boldsymbol{x} \;=\; 0,$$

where

$$\boldsymbol{a}_\ell = (a_{\ell_0}, a_{\ell_1}, \ldots, a_{\ell_n})^{\mathrm{T}}; \quad a_{\ell_0} \in \mathbf{R}^*, \quad a_{\ell_\kappa} \in \mathbf{R}\ (\kappa \in \{1, 2, \ldots, n\}).$$

Then $\boldsymbol{x}$, the coordinates of a point on both hyperplanes μ and ν (hence, the point is in the $(n-2)$-dimensional intersection subspace of the two hyperplanes), satisfies

$$\alpha_\mu (\boldsymbol{a}_\mu \cdot \boldsymbol{x}) + \alpha_\nu (\boldsymbol{a}_\nu \cdot \boldsymbol{x}) \;=\; 0, \tag{4.2}$$

where α_ℓ ($\ell = \mu, \nu$) are real numbers. By fixing the values of α_ℓ so that $\boldsymbol{c}$ satisfies (4.2), we obtain the interpretation hyperplane (in $\mathbf{P}^n$) of the intersection subspace of two hyperplanes μ and ν:

$$(a_{\nu_0}\boldsymbol{a}_\mu - a_{\mu_0}\boldsymbol{a}_\nu) \cdot \boldsymbol{x} \;=\; 0.$$

Therefore, $a_{\nu_0}\boldsymbol{a}_\mu - a_{\mu_0}\boldsymbol{a}_\nu$ is the homogeneous coordinates of the interpretation hyperplane of the intersection subspace of hyperplanes μ and ν; $F_{P_*^{-\mathrm{T}}}(a_{\nu_0}\boldsymbol{a}_\mu - a_{\mu_0}\boldsymbol{a}_\nu)$ is the interpretation vector of the intersection subspace. It is important to note that we have indeterminacy of a scaling factor between the vector $F_{P_*^{-\mathrm{T}}}(a_{\nu_0}\boldsymbol{a}_\mu - a_{\mu_0}\boldsymbol{a}_\nu)$ and the vector $\boldsymbol{n}_{\mu\nu}$ we actually obtain

[3]For a square matrix P, $P^{-\mathrm{T}}$ is $(P^{\mathrm{T}})^{-1}$ or equivalently $(P^{-1})^{\mathrm{T}}$.

as a result of observing the subspace. Therefore, defining

$$a_{\mu\nu} = F_{P_*^{-T}}\left(a_{\nu_0} a_\mu - a_{\mu_0} a_\nu\right),$$

we have

(4.3) $$n_{\mu\nu} = \rho_{\mu\nu}\, a_{\mu\nu} \quad (\rho_{\mu\nu} \neq 0).$$

Here, $\rho_{\mu\nu}$ is a scaling factor and its value is not known. In line with treating $n_{\mu\nu}$, we define an $n \times n$ matrix $M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ $(k, \ell \in \{1,2\})$ as a counterpart of $N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$:

(4.4) $$M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} := \Big[\, a_{Lk_1 Lk_2} \,|\cdots|\, a_{Lk_{i-1} Lk_i} \,|\, a_{Lk_i C_1} \,|\, a_{C_1 C_2} \,|\cdots|$$
$$a_{C_{n-i-j} C_{n+1-i-j}} \,|\, a_{C_{n+1-i-j} R\ell_j} \,|\, a_{R\ell_j R\ell_{j-1}} \,|\cdots|\, a_{R\ell_2 R\ell_1} \Big].$$

(4.3) and (4.4) yield

(4.5) $$\det N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = P_{k\ell} \cdot \det M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}},$$

where

$$P_{k\ell} := \rho_{Lk_i C_1} \cdot \rho_{C_{n+1-i-j} R\ell_j} \cdot \prod_{\kappa \in \Omega_{Lk} - \{Lk_i\}} \rho_{i_\kappa, i_{\kappa+1}} \cdot \prod_{\kappa \in \Omega_C - \{C_{n+1-i-j}\}} \rho_{i_\kappa, i_{\kappa+1}} \cdot \prod_{\kappa \in \Omega_{R\ell} - \{R\ell_j\}} \rho_{i_{\kappa+1}, i_\kappa}.$$

We again attach $'$ (prime) to the notations above in the case where an admissible transformation has operated on S. Hence, we obtain[4]

$$\text{LHS of (3.2)} = \frac{\det M_{\Omega_{L1},\Omega_C,\Omega_{R1}} \cdot \det M_{\Omega_{L2},\Omega_C,\Omega_{R2}}}{\det M_{\Omega_{L1},\Omega_C,\Omega_{R2}} \cdot \det M_{\Omega_{L2},\Omega_C,\Omega_{R1}}},$$

$$\text{RHS of (3.2)} = \frac{\det M'_{\Omega_{L1},\Omega_C,\Omega_{R1}} \cdot \det M'_{\Omega_{L2},\Omega_C,\Omega_{R2}}}{\det M'_{\Omega_{L1},\Omega_C,\Omega_{R2}} \cdot \det M'_{\Omega_{L2},\Omega_C,\Omega_{R1}}}.$$

Now, to prove Theorem 3.1 it suffices to show the following lemma (applying the results of four combinations of k and ℓ in Lemma 4.1 to the two equations above, completes the proof of Theorem 3.1).

Lemma 4.1 Suppose that a point (with coordinates $\boldsymbol{x}$) in S changes its coordinates to $\boldsymbol{x}'$ after an admissible transformation T $(\in \mathcal{T}_S)$ as follows:

$$\lambda\, \boldsymbol{x}' = T\, \boldsymbol{x} \qquad (\lambda \in \mathbf{R}^*).$$

[4]LHS and RHS stand for the left-hand side and the right-hand side, respectively.

Let $\mathrm{rank} M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n$, where $k, \ell \in \{1,2\}$. Then, we have, independent of f_{P_*},

$$\text{(4.6)} \qquad \mathrm{rank} M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n,$$

$$\text{(4.7)} \qquad \det T \cdot \prod_{\kappa \in \Gamma_{k\ell}} a_{\kappa 0} \cdot \det M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = \lambda^{n+1} \cdot \prod_{\kappa \in \Gamma_{k\ell}} a'_{\kappa 0} \cdot \det M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}},$$

where

$$\Gamma_{k\ell} := \Omega_{\mathrm{L}k} \cup \Omega_{\mathrm{C}} \cup \Omega_{\mathrm{R}\ell} - \{\mathrm{L}k_1, \mathrm{R}\ell_1\}.$$

Proof: It follows from the definition of $M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$ that

$$\text{(4.8)} \qquad \det M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = \frac{1}{\det P_*} \cdot (-1)^{\mathrm{Mod}_2(n)} \cdot \prod_{\kappa \in \Gamma_{kl}} a_{\kappa 0} \cdot \det A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}.$$

Here, $(n+1) \times (n+1)$ matrix $A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}$ is defined by

$$A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} := [\, a_{\mathrm{L}k_1} \mid a_{\mathrm{L}k_2} \mid \cdots \mid a_{\mathrm{L}k_i} \mid a_{\mathrm{C}_1} \mid \cdots \mid a_{\mathrm{C}_{n+1-i-j}} \mid a_{\mathrm{R}\ell_j} \mid \cdots \mid a_{\mathrm{R}\ell_1} \,];$$

and, for a natural number n, Mod_2 is a function such that

$$\mathrm{Mod}_2(n) = \begin{cases} 0 & n\text{: even} \\ 1 & n\text{: odd.} \end{cases}$$

Similarly, for after admissible transformation T, we define $(n+1) \times (n+1)$ matrix

$$A'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} := [\, a'_{\mathrm{L}k_1} \mid a'_{\mathrm{L}k_2} \mid \cdots \mid a'_{\mathrm{L}k_i} \mid a'_{\mathrm{C}_1} \mid \cdots \mid a'_{\mathrm{C}_{n+1-i-j}} \mid a'_{\mathrm{R}\ell_j} \mid \cdots \mid a'_{\mathrm{R}\ell_1} \,],$$

then we obtain

$$\text{(4.9)} \qquad \begin{aligned} \det M'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} &= \frac{1}{\det P_*} \cdot (-1)^{\mathrm{Mod}_2(n)} \cdot \prod_{\kappa \in \Gamma_{kl}} a'_{\kappa 0} \cdot \det A'_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} \\ &= \frac{1}{\det P_*} \cdot (-1)^{\mathrm{Mod}_2(n)} \cdot \frac{\lambda^{n+1}}{\det T} \cdot \prod_{\kappa \in \Gamma_{kl}} a'_{\kappa 0} \cdot \det A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}}, \end{aligned}$$

since

$$a'_i = \lambda T^{-\mathrm{T}} a_i.$$

Note that $a'_{i_0} \neq 0$ is satisfied since $T \in \mathcal{T}_S$. (4.8) and (4.9) immediately yield (4.7). It is clear that (4.7) is independent of f_{P_*}.

Since $a_{\kappa 0} \neq 0$ $(\kappa \in \Gamma_{kl})$, it follows from (4.8) that $\mathrm{rank} M_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n$ is equivalent to $\mathrm{rank} A_{\Omega_{\mathrm{L}k},\Omega_{\mathrm{C}},\Omega_{\mathrm{R}\ell}} = n+1$. Then we have (4.6) from (4.9) since $\lambda \neq 0$ and $a'_{\kappa 0} \neq 0$ $(\kappa \in \Gamma_{kl})$. □

Remark 4.1 (4.8) shows that $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ is equivalent to $\det M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ $(k, \ell \in \{1,2\})$. Namely, n subspaces in $\mathbf{P}^n - \{c\}$ (the intersection subspaces of the adjacent hyperplanes in $\Omega_{Lk} \cup \Omega_C \cup \Omega_{R\ell}$) share a common point in $\mathbf{P}^{n-1}$ through the projection f_{P_*} iff $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} = 0$ (see Observation 4.1 below). We assume that $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} \neq 0$ is satisfied by $(n+1)$ hyperplanes that determine these n subspaces (intuitively, this assumption is equivalent to the random choice of $(n+1)$ hyperplanes). Moreover, Lemma 4.1 indicates that if $\det A_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}} \neq 0$ holds, we can guarantee that these n subspaces after any admissible transformations never share a common point in $\mathbf{P}^{n-1}$ through the projection f_{P_*}. □

We now turn to the proof of Theorem 3.2. From (3.3) it is easy to see that Inv_{ij} is nonsingular iff the values of the determinants of $N_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ $(k, \ell \in \{1,2\})$ are not zero. Hence, the necessary and sufficient condition under which Inv_{ij} is nonsingular is that the values of the determinants of $M_{\Omega_{Rk},\Omega_C,\Omega_{R\ell}}$ are not zero (see (4.5)). Observation 4.1 below indicates that when n subspaces of $(n-2)$ dimensions in $\mathbf{P}^{n-1}$ do not share a common point, the value of the determinant of $M_{\Omega_{Lk},\Omega_C,\Omega_{R\ell}}$ is never zero. This argument yields Theorem 3.2 (see Remark 4.1).

Observation 4.1 Let n different subspaces i $(i = 1, 2, \ldots, n)$ of $(n-2)$ dimensions in $\mathbf{P}^{n-1}$ be

$$\sum_{k=0}^{n-1} a_{i_k} X_k = 0,$$

where

$$\sum_{k=0}^{n-1} {a_{i_k}}^2 \neq 0.$$

They do not share a common point in $\mathbf{P}^{n-1}$ iff

$$\det \begin{bmatrix} a_{1_0} & \cdots & a_{i_0} & \cdots & a_{n_0} \\ a_{1_1} & \cdots & a_{i_1} & \cdots & a_{n_1} \\ \vdots & & \vdots & & \vdots \\ a_{1_k} & \cdots & a_{i_k} & \cdots & a_{n_k} \\ \vdots & & \vdots & & \vdots \\ a_{1_{n-1}} & \cdots & a_{i_{n-1}} & \cdots & a_{n_{n-1}} \end{bmatrix} \neq 0.$$

□

5 Conclusion

We have investigated the existence of projective invariants under the condition that the projection from $\mathbf{P}^n - \{c\}$ to $\mathbf{P}^{n-1}$ is achieved by an element of $\mathcal{F}$, and the class of admissible transformations is $\mathcal{T}_S$ where S is the set whose elements are $(n-2)$-dimensional subspaces in $\mathbf{P}^n - \{c\}$. Then, for given integers i and j such that $1 \leq i \leq j \leq n-i$, we derived projective invariant, independent of the element of $\mathcal{F}$, Inv_{ij} in (3.3) from $(n+i+j)$ subspaces of $(n-2)$ dimensions, where these subspaces are the intersections of the adjacent hyperplanes of $(n+i+j+1)$ hyperplanes arranged in the letter H. Accordingly, the $(n+i+j)$ subspaces are also arranged in the letter H (hence, the $(n+i+j)$ subspaces could n-dimensionally exist). Note that the number of subspaces in the left-upper part of H is i, whereas that in the right-upper part is j; and the arrangement is symmetrical with respect to the horizontal part. Let us remark again that the horizontal part could be empty since $i+j=n$ is possible. In addition, the number of this kind of invariants in $\mathbf{P}^n - \{c\}$ is $\lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor\right)$.

Furthermore, the nonsingularity condition for Inv_{ij}, i.e., the necessary and sufficient condition making Inv_{ij} nonsingular, was given. Inv_{ij} is nonsingular iff (COND) below is satisfied by n subspaces among the $(n+i+j)$ subspaces, i.e., n aligned intersection subspaces of the adjacent hyperplanes, which include the horizontal part Ω_C of H, in the arrangement H above (we always have four cases).

(COND) Not singular is an $(n+1) \times (n+1)$ matrix whose column vectors are the homogeneous coordinates (in $\mathbf{P}^n - \{c\}$) of $(n+1)$ hyperplanes that determine n subspaces of $(n-2)$ dimensions.

The nonsingularity condition guarantees that Inv_{ij} is not only well-defined but nondegenerated; it also ensures that the values of Inv_{ij} are numerically stable when they are calculated in practical situations. We should remark that this condition is almost always satisfied when we randomly choose $(n+i+j+1)$ hyperplanes in $\mathbf{P}^n - \{c\}$.

Elaboration of investigating the existence of projective invariants under another projection class is left open for future research.

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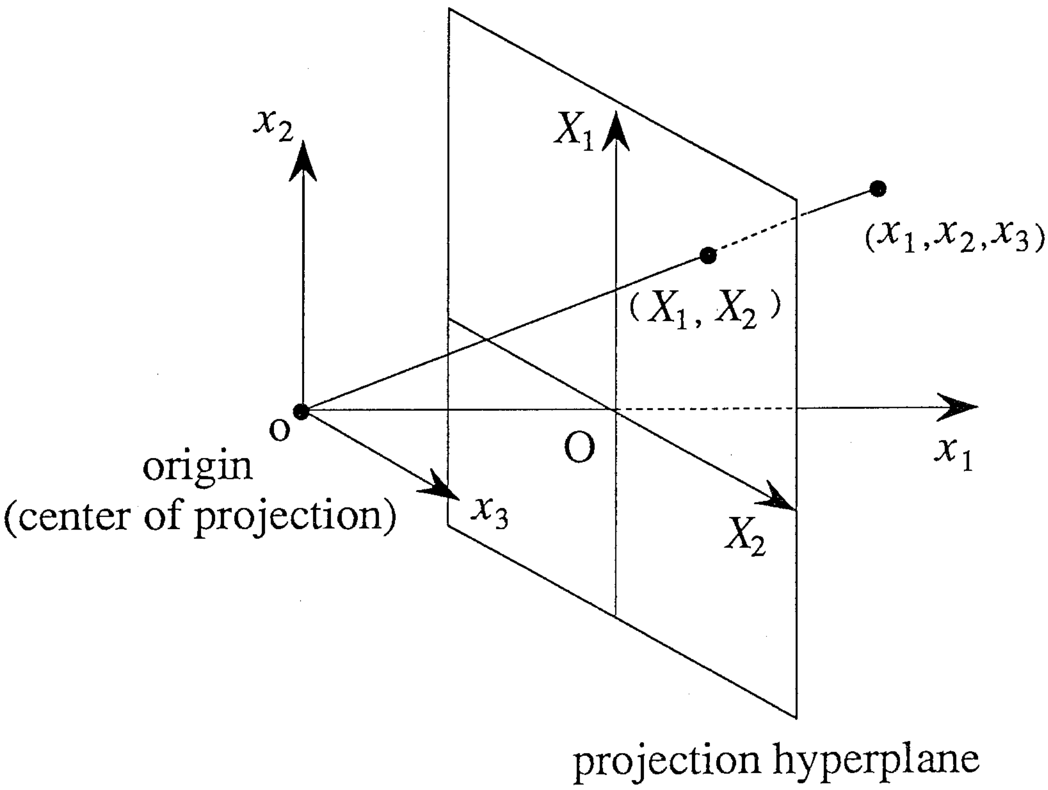

Fig. 1: Central projection attached at origin O ($n = 3$)

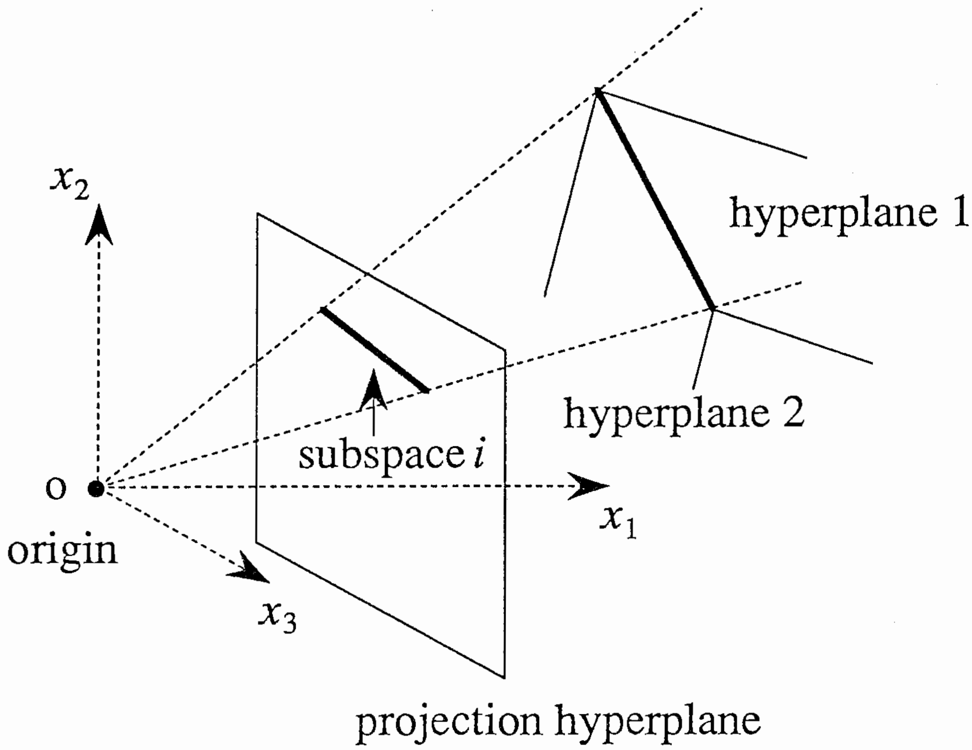

Fig. 2: $(n-2)$-dimensional subspace determined as a pair of hyperplanes $(n=3)$

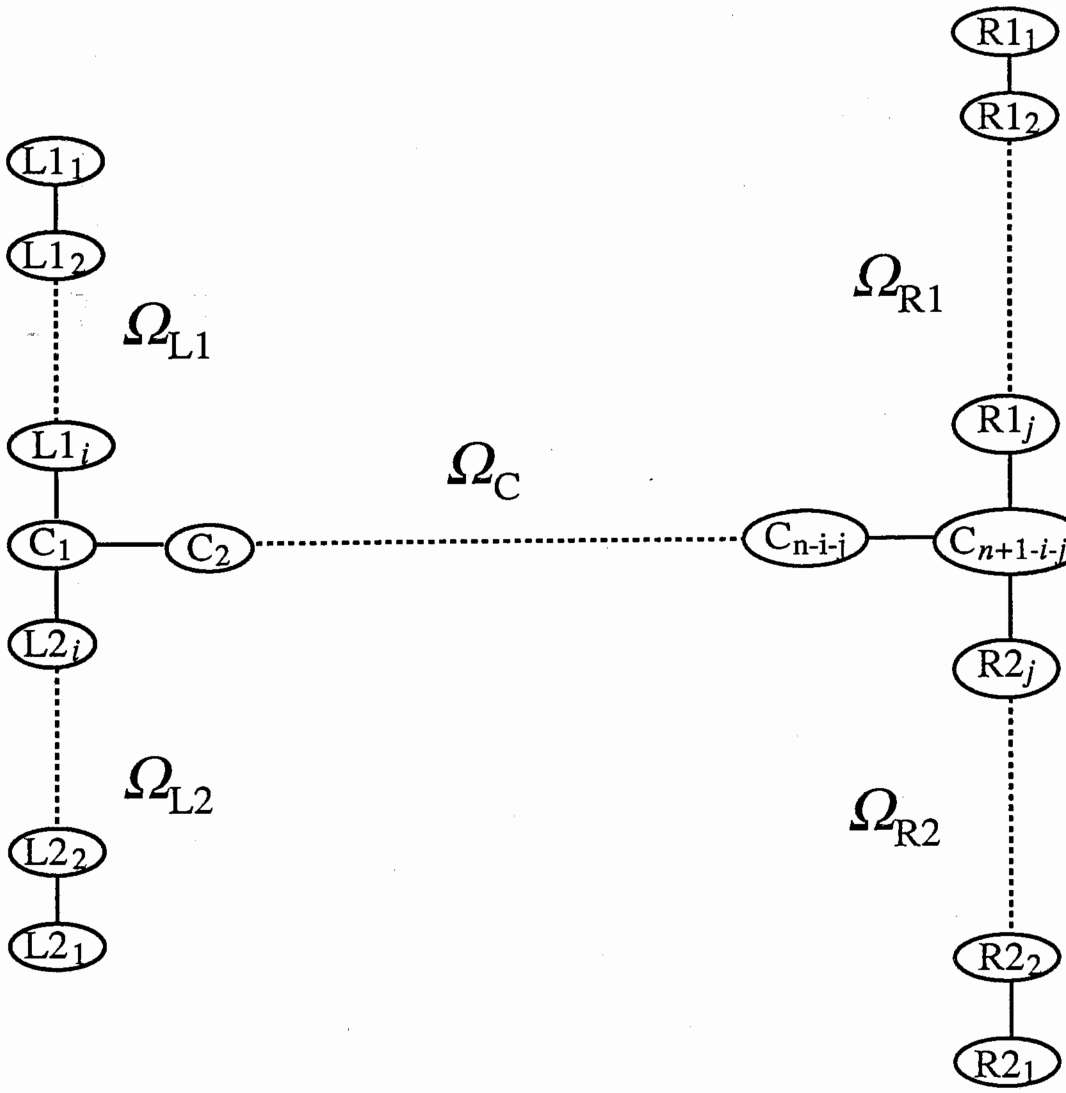

Fig. 3: Arrangement H of $(n + i + j + 1)$ hyperplanes and $(n - 2)$-dimensional subspaces as the intersections of the adjacent hyperplanes (the numbers in ellipses represent hyperplanes; the lines and the dashed lines represent $(n - 2)$-dimensional subspaces)

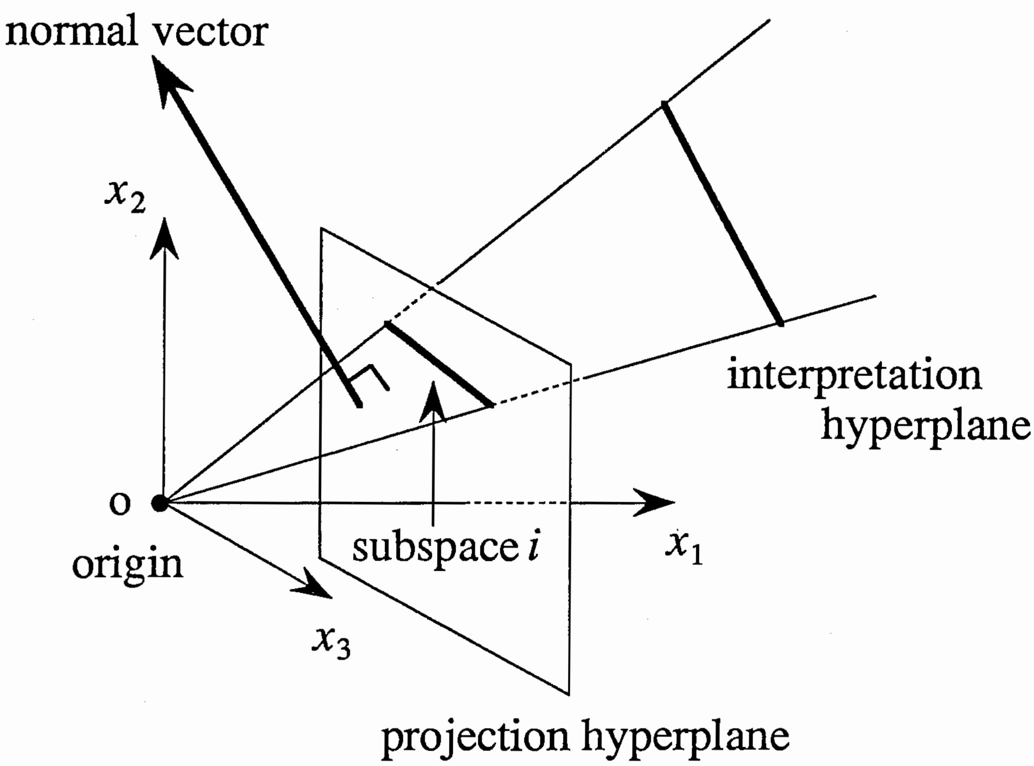

Fig. 4: Subspace i and the homogeneous coordinates (or equivalently the normal vector) of its interpretation hyperplane ($n = 3$)