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Real Time Hand Motion Detection and Recognition

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Abstract

Low level visual techniques for hand motion detection and hand gesture recognition are typically performed on a series of static images and are computationally intensive. The dynamic nature of hand motion and the requirements for real time processing place further demands upon visual sensing. Simple hand gestures such as pointing, grasping, or releasing may, however; be viewed as temporal patterns of spatially coherent motion. The spatial and temporal coherency of the motion pattern is due to the systematic changes in the relationships among the individual fingers and the hand. The common motion of the fingers toward the center of the hand is one example of coherent motion. Dynamical systems which mimic the coherency of hand motion provide an alternative to traditional computer vision approaches for hand gesture recognition.

A completely new approach to hand motion detection and recognition is proposed in this project. This approach is based upon analog computation and uses dynamical systems concepts to design a novel visual sensor which can detect and recognize a fixed repertoire of dynamic hand motions. The real time performance of this sensor is guaranteed by the fact that the dynamical systems can be directly implemented as electronic circuits. The design of this hand motion sensor comes in three parts: (a) mathematical analysis, (b) simulation of the dynamics, and (c) analog implementation. Research at the begining of the project will focus on the design, analysis, and simulation of the proposed sensor. Later work will focus on the analog implementation.

Contents

1 Introduction

The visualization of complex information has become an increasingly common activity in modern society. Recent advances in the technology of scientific visualization have fostered the need for simpler, more humanistic methods for interacting with the computer generated visualization. One mode of human interaction with the visualization is through voice commands to the computer. Another mode of interaction occurs through visual sensing of hand movements and eye position to infer the person's intent. An illustrative example of the interdependent relationship between language and vision has emerged from recent studies of sign language[1]. Sign language is a form of communication in which meaning is conveyed through a series of complex hand gestures and motions which follow specific linguistic rules. The combination of hand motions and linguistic rules may be viewed as a highly complex system in which the dynamics of the hand motions are governed by the linguistic rules. This observation, if it is correct, suggests that the sensing of hand motion may also be governed by a dynamical system.

The set of hand gestures required for interaction with a computer visualization forms a simpler subset of the gestures than those found in sign language. The theme of this research project is the application of dynamical systems concepts to the design of a novel visual sensor which can detect and recognize a fixed repertoire of hand motions. The real time performance of this sensor is guaranteed by the fact that the dynamical systems can be directly implemented as electronic circuits. The design of this hand motion sensor comes in three parts: (a) mathematical analysis, (b) simulation of the dynamics, and (c) analog implementation. Research at the beginning of the project will focus on the design, analysis, and simulation of the proposed sensor. Later work will focus on the analog implementation.

2 Relation of Dynamics to Hand Motion Recognition

The information afforded by a sequence of images containing objects moving at different speeds and directions is dynamic. Any mechanism for processing this visual information must be able to respond in real time within a constantly changing environment. Yet computer vision has traditionally tried to abstract away from the dynamic nature of real time processing using various strategies such as reducing intensity images to simple zero crossings or by discretizing time and searching for correspondences between frames. An alternative approach for representing image information involves the building of dynamics directly into the computational mechanisms. The basic idea of this approach is to first map dynamic information from the image sequence onto a model realized as a (possibly nonlinear) dynamical system. Subsequent information processing, such as feature detection, feature linking, ambiguity resolution, or perceptual grouping occurs through the direct interactions among the dynamical systems.

The introduction of some terminology from dynamical systems theory may be helpful at this point. A dynamical system in its most general form is a system of ordinary differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1}$$

where the vector $\mathbf{x} = (x_1, \cdots, x_n)$ indicates the variables in the system and the function $f(\mathbf{x})$ describes the dynamical behavior of each variable as well as the interdependencies among all of the system variables.

The information processing capabilities of a dynamical system depend upon the local behavior of *attractors*, or limit sets, in the state space of the system. Each attractor is a set of points toward which the dynamical system tends to converge. A limit set containing a single point is called a *fixed point attractor*, since any initial state of the dynamical system in the neighborhood of this point will cause the system to converge onto a single, fixed point. A limit set containing a closed trajectory is called a *limit cycle* since any initial state near this closed trajectory, or cycle, will yield a trajectory which eventually converges onto this limit set.

In this research project, we shall associate each hand gesture with the trajectory along a limit cycle attractor in the dynamical system. The image processing for hand gesture recognition occurs at two levels: 1) *sensory dynamics*, and 2) *recognition dynamics.* According to this approach, visual processing is performed in parallel at each location in an image by multiple, relatively simple (sensory) dynamical systems embedded within the single complex (recognition) system. In this way, multiple vision tasks can be unified within a single complex system. For example, features may be identified with limit sets of a multi-attractor system. The position of the feature can be obtained by mapping the profile of the $\nabla^2 G$ (the Laplacian of a Gaussian) image function[2] onto a limit cycle attractor where phase along the limit cycle corresponds to relative image position. Velocity information can be recovered from the temporal derivative of the $\nabla^2 G$ operator[3]. The computation of motion for an intensity edge passing through a small window in the image can be achieved by the design of a dynamical system which maps the position and motion profiles onto a two-dimensional submanifold of the model dynamics. Spatial interactions among dynamical systems are driven by concurrent inputs from the local gray-level context of the image features and the rate of change of intensity in a temporal sequence of images. Concurrent inputs are important for driving multiple variable systems along the surface of a manifold toward a stable attractor.

The potential for the application of dynamical systems to solve many low level vision problems exists, however the synthesis of complex nonlinear dynamical systems from simple ones remains as an open research problem. The use of normal forms facilitates this effort by transforming complicated dynamical systems into an equivalent system in a standard form. In this way, it is possible to unify the design and synthesis of many different systems.

The method for recognizing hand gestures will be an extension of methods previously developed for detecting low level motion of intensity edges[4]. Low level motion detection was accomplished by mapping the phase information from a pair of Gabor filters in phase quadrature onto the phase of a limit cycle attractor in the model dynamics. As a result of this mapping, the dynamical system rapidly converged to a stable limit cycle provided the temporal evolution of profiles generated by the Gabor filters closely matched the phase and speed of some trajectory near the limit cycle of the dynamical system. If the object motion is not commensurate with the model dynamics, then control parameters can be adaptively modified to improve the match between the sensed object motion and the limit cycle trajectory. According to this paradigm, visual signals derived from the time-varying image are mapped onto specific variables in the model dynamics and the tracking of objects with different image velocities is achieved through the adaptive control of parameters in the model dynamics.

The motivation for using dynamical systems to perform the hand gesture recognition task is derived from several sources:

1. The requirements for real time processing of increasingly complex hand gestures.

2. The need to integrate the visual modules such as feature detection, motion analysis, and image segmentation.
3. The desire to provide the capability for other sensory modalities (such as speech) to influence the performance of the hand motion sensor.

Integration of these visual modules is important for improving the robustness of the information processing tasks of the vision system[5, 6, 7]. To achieve this integration it is first necessary to synthesize simple dynamical systems which perform specialized tasks as outlined above. Then techniques must be developed to embed these simple systems into a common dynamical system.

The use of dynamical systems also facilitates the incorporation of other sensory modalities into the sensing process. This can be accomplished, for example, by providing a speech recognition system with knowledge of the dynamical behavior of the hand motion sensor. The speech system could then alter the dynamics in prescribed ways in order to enhance the detection of certain hand gestures (eg. pointing) when that kind of behavior (eg. pointing) is anticipated from the speech input. In this way the speech system could influence the operation of the vision system without requiring specific knowledge concerning the internal processing occurring within the vision system.

3 The Hand Recognition Project

A completely new approach to hand motion detection and recognition is proposed in this project. The hand motion sensor is to be designed as an electronic circuit from its earliest inception in order to insure real time performance. It is highly significant for this project that the earliest designs be created as electronic circuits rather than computer algorithms in order that the unique properties of electronic circuits be accounted for in *all* design decisions.

The advancement of this project depends upon mathematical results currently under active investigation in the field of *normal form analysis of nonlinear dynamical systems*. In the course of developing the hand motion sensor, new mathematical results will have to be derived and existing theory extended to more complex, real world problems. Thus, significant contributions to this field of mathematics will be derived from this project.

Since both the approach is new and the mathematics is actively being developed, the initial problems addressed in this research will be very simple. Once the basic principles have been developed for the simplest cases, then more complex detection and recognition tasks will be investigated.

3.1 Problem formulation

Simple hand gestures such as pointing, grasping, or releasing may be viewed as patterns of coherent motion in both space and time. The coherency of the motion pattern is due to systematic changes in the relationships among the individual fingers. Consider, for example, the grasping gesture in which all fingers move inward toward the center of the hand. In the hand gesture for releasing all fingers move outward.

The problem of visually sensing the motion of a hand is one example of a nonlinear, dynamic process requiring context dependent integration of multiple sources of visual information. The sources of visual information include; (a) the *position* of the hand in space, (b) the *configuration* of the fingers, and (c) the *motion* of the hand while the gesture is expressed. The integration of information from the hand and finger motions depends upon the context established by the three aspects of position, configuration, and motion. The recognition of hand gestures is a dynamic process since temporal changes in the configuration of the hand must be considered. The detection problem is nonlinear since discrimination among different gestures by the same dynamical system is required. For these reasons, *nonlinear dynamical systems* will be used to design a visual sensor for hand gesture recognition.

3.2 Goals of this research

The fundamental goal of this research is to develop a sensor for the real time recognition of simple hand gestures. In the course of developing this sensor, it is hoped that the following tasks will be accomplished:

1. Extend existing hand gesture recognition approaches to include dynamic motion.
2. Develop mathematical theory necessary for analysis and design of sensors for complex motion which include partial occlusion of the hand.
3. Implement a prototype analog device for performing the recognition task.
4. Provide facilities for an external system, such as speech, to influence the dynamical behavior of the sensor.

3.3 Scope

In this project, the basic approach proposed for investigating the dynamic processes of hand motion is to map the trajectories obtained from the hand and finger motion onto stable structures of the model dynamics. The initial research will focus on the use of the finger tips and center of the hand as visual inputs to a single dynamical system with either one or multiple attractors. The long term goal is to use a spatial array of these dynamical systems to provide additional processing through subsequent interactions among the array elements. Thus the motion of discrete features will be investigated first. These results can be applied later to investigate the continuous motion of smooth surfaces.

3.4 Representation and method

The instantaneous position of each finger tip will be represented by a 3-D vector (x, y, z). The position of the hand is also represented by a 3-D vector, thus the hand gesture is represented in an 18-D space plus the continuous time variable. The dynamic motion of the hand gesture is to be represented by attractors in the system

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{2}$$

where the function $f(\mathbf{x})$ describes the dynamics of the sensor. Each hand gesture is determined by the motion of the hand from some initial state toward some final configuration. Each gesture is represented by the temporal trajectories of 18 variables of the dynamical system.

For monocular images only two components of each feature (x, y) are available from the image sequence. The dynamics of the attractor force the third component z to be consistent with the other two. In dynamical systems terminology, each gesture is represented by a basin of attraction in a multi-attractor system. Thus the image inputs (I_{x_i}, I_{y_i}) drive the variables (x_i, y_i, z_i) from some initial state toward the basin of attraction corresponding to that particular hand gesture.

Modeling of simple motion as bifurcations in low dimensional dynamical systems has been suggested by various researchers[8, 9, 10]. A paradigm for the study of nonlinear systems has recently emerged with the introduction of Chua's circuit[11, 12, 13, 14] and its associated canonical circuit family[15]. The relative simplicity of Chua's circuit provides a convenient model for the dynamics and bifurcation phenomena in more complex systems. Such nonlinear systems have always played an important role in the study of natural phenomena, but only recently have advances in dynamical systems theory[16, 17] and the development of software for performing symbolic computations[18] made the investigation of complex nonlinear systems feasible.

3.5 Advantages of the method

One desirable property of a system is that the tasks to be performed are initially quite simple and progressively increase in complexity. However, current approaches for real time hand gesture recognition cannot be generalized to work with more complex gestures since each additional task to be performed in the recognition process requires more time or greater computational resources. Thus the extension from simple gestures to complex gestures can only be achieved by increasing temporal delays or by sharply increased equipment cost. Perhaps the greatest potential advantage of an analog approach to hand gesture recognition is that progressively more complex recognition tasks can be performed by incrementally increasing the complexity of the analog circuits. The analog approach, however, is not without its own unique set of problems[19]. The primary advantages expected from this approach are;

1. Real time performance is guaranteed since all processing is implemented in analog form.

2. Partial occlusion of the hand during the formation of the gesture should result in a graceful degradation of the recognition task.

3. Extension to the recognition of more complex hand gestures should occur in constant time; however, the complexity of the initial design may be greatly increased.

4. External control of the sensor should be possible by knowing only the dynamics of the sensor and not the data being processed. This leads to a modular design in which modules can interact with each other without knowing the internal operation of the other modules.

An additional, but less tangible, advantage of this method is that completely new approaches for hand gesture recognition and sign language recognition may be developed which would not have been possible using the traditional methods.

3.6 Disadvantages of the method

The special properties of nonlinear dynamical systems facilitate certain tasks, however, other tasks remain very difficult. Furthermore, the mathematics required to understand nonlinear systems is still incomplete, thus making the resolution of these tasks more difficult. For example, the understanding of low order autonomous systems is well developed, however, the use of external signals to drive the dynamical system may create very complex behavior which can not be completely analyzed.

The primary disadvantages of the dynamical systems approach to hand gesture recognition are:

1. Complexity of the initial design is very high.

2. Dependency upon mathematical techniques from an area currently under active research increases the difficulty of the project.

3. Interpretation of the output of a complex dynamical system may be nearly as difficult as the original task.

4. Lack of flexibility (compared to computer programs) due to the hardware implementation and mathematical complexity.

5. The complexity of the dynamical systems required to achieve acceptable performance may exceed implementation technologies.

It is hoped that the advantages provided by the use of dynamical systems will far exceed the difficulties listed above. Further research and the application of this method to practical problems are needed to resolve this issue.

3.7 Problems to be solved

Early experiments will emphasize the design of the dynamical systems, thus simulated or drastically simplified visual inputs will initially be used. The long term objective is to use inputs from the image sequence which are obtained at as low a level of processing as possible. Thus three different types of inputs will be used depending upon the level of sophistication of the sensor design. These inputs are;

1. Gabor filters — a pair of filters in phase quadrature which respond to inputs from a fixed location in the image. Useful for generating simple signals for motion detection and for testing a spatial array of dynamical systems.

2. Coordinates of image feature — the (x, y) coordinates of simple features provide a simple method for providing inputs for a multi-attractor system. Useful for detecting large motions with a single dynamical system.

3. Low-level inputs — the Laplacian of a Gaussian and its time derivative provide meaningful position and velocity information while requiring a minimum of preprocessing. Useful for investigating the cooperative behavior of a large array of motion sensors in which each sensor detects motion in a small portion of the image.

The problems in this proposal are starred to indicate the level of difficulty. The rating system is, roughly;

- ⋆ Simple application of existing mathematical theory.
- ⋆⋆ Slightly more subtle problem requiring the development of new computer code to extend existing mathematical theory to more complex conditions. Should normally require one or two weeks to complete.
- ⋆⋆⋆ A much harder problem requiring the extension of existing theory and the writing of new code. Could take two months.
- ⋆⋆⋆⋆ An even harder problem that could require four months to solve.
- ⋆⋆⋆⋆⋆ An open research question of unknown complexity.

The experiments described in this section are intended to be conducted sequentially starting with relatively simple tasks and proceeding to more complex ones.

Motion detection

Goal is to detect the occurrence of motion without recognizing the velocity and direction of the motion.

- ⋆ Detect the motion of an ideal edge through a fixed window in the image. Convolve the image with Gabor filters and use the filter outputs to drive the dynamical system onto a limit cycle.
- ⋆⋆ Detect the circular motion of one finger tip by mapping the (x, y) position of the finger onto the dynamical system.
- ⋆⋆ Use a spatial array of 32×32 identical dynamical systems of the scroll type to detect the occurrence of motion in the image. This involves mapping the outputs from an array of Gabor filters onto the corresponding elements in the spatial array. Synthetic inputs will be used to evaluate the performance of the motion detection array.
- ⋆⋆ Run the simulation for the motion detection array with real world inputs obtained with the Pipe processor to implement the image filtering.

The objective of these experiments is to demonstrate that hand motion can be detected using a spatial array of identical dynamical systems as the sensing device. No effort is made to recognize specific motions or hand gestures.

Trajectory tracking

The goal of this series of experiments is to develop methods for detecting different types of motion. Each type of motion constitutes a different trajectory in 3-D space. Two approaches for detecting these trajectories will be investigated. In the first approach the form of the dynamical system remains fixed, but the transformation of the sensed image inputs changes for each different trajectory to be detected. In the second approach, the form of the dynamical system is altered to detect different trajectories, while the filtering of the sensed image remains unchanged.

- ⋆⋆ Mathematically evaluate the use of coordinate transformations to transform input trajectories into a form appropriate for the existing dynamical system.
- ⋆⋆ Mathematically evaluate the use of potential functions added to the dynamical system to change the path of the limit cycle trajectory. A critical problem which may arise here is that subtle changes in the shape of the attractor may strongly influence the domain of attraction or the way that the system responds to external inputs.
- ⋆ Coordinate transformations probably will not work since they must be time varying. So the first approach will most likely not be feasiable.
- ⋆⋆ Test a variety of modifications to the dynamics with simulated and real visual inputs.
- ⋆⋆ Implement a spatial array of the modified dynamical systems such that each element receives (x, y) position of a single feature point moving in the image. A large overlap of the input fields for each array element should be used to test the preferental response of those elements with strong inputs over those elements where the inputs are only partially within their respective fields.
- ⋆⋆ Repeat the trajectory tracking experiment with modified dynamical systems to detect several different trajectories.

The use of coordinate transformations to map the visual input onto an existing dynamical system is expected to require a time varying transformation. Hence that approach will not be practical for the intended applications. The extent to which the existing dynamical system can be modified without qualitatively changing its behavior may be rather limited due to nonlinearities in the dynamics. Some combination of the two methods will probably be required in subsequent developments.

Trajectory discrimination

Whereas the previous experiments attempted to detect a single trajectory in space, this series of experiments will attempt to discriminate among several different possible trajectories. This will require the extension of the previous results for single attractor systems to multiple attractor systems. The use of dynamical systems with multiple attractors will greatly increase the complexity of the analysis required to design desired behaviors into the dynamics. Many of the problems encountered in this section are open research questions. We shall attempt to address only the simpler of these.

- ⋆⋆⋆ Extend the normal form analysis of single attractor systems to multi-attractor systems.
- ⋆⋆⋆⋆ Extend the theory for the alteration of stable trajectories from the single attractor case to multi-attractor case. The extension from one attractor to two attractors should be straightforward. The extension to higher order systems will become income increasingly complex due to the interactions among attractor domains.

- ⋆⋆ Run computer simulations of the coupled dynamical systems to evaluate the theoretical analysis. Once the analysis is complete, the simulation is simple. Without a detailed analysis, however, simulation would be impossible since there would be no way to know how to design or test the dynamics.
- ⋆⋆⋆⋆ Implement the coupled dynamical systems as analog circuits and verify experimentally the simulation results. This will be a time consuming project since the equipment to automate the control, the generation of input signals, and the testing of the analog circuits will all have to be designed and assembled.

Since the overall goal of the project is real time hand gesture recognition it is important to implement the methods using electronic circuits. The hand motion detection and recognition approach developed in this project is specifically designed for analog implementation to guarantee real time performance. The availability of analog circuits will also greatly facilitate testing since computer simulations can be quite slow especially when many different conditions must be tested.

Learning new trajectories

The ability to automatically learn new trajectories for hand gestures is important for improving the flexibility of the hand motion sensor. The following series of experiments attempts to develop a method for learning new trajectories for the dynamical systems.

- ⋆⋆⋆⋆ Learn an arbitrary trajectory for a dynamical system with a single attractor.
- ⋆⋆⋆⋆⋆ Learn an arbitrary trajectory for a dynamical system with multiple attractors.

These problems are currently open research questions. One approach to the problem of learning new trajectories follows from the observation that limit cycle trajectories exist on the surface of the center manifold of the dynamical system. Therefore, the learning of new trajectories could potentially be achieved by perturbing the surface of the center manifold by the operations of stretching, compressing, or folding of the surface to fit the desired trajectory.

3.8 Future plans: Real time hand motion recognition system

The previous sets of experiments was designed to develop the essential tools for; (a) mapping trajectories onto dynamical systems, (b) modifying trajectories, and (c) learning new trajectories. The design of a complete recognition system involves the integration of these techniques with the necessary hardware to provide real time performance. Future plans for creating a real time hand motion recognition system include the following:

- ⋆⋆⋆⋆ Develop a graphical cartoon hand to simulate the action of the hand in a virtual reality environment. The current state of the dynamical system can be used to control the configuration of the simulated hand.
- ⋆⋆⋆⋆ Use speech input to assist the hand recognition sensor and to modify its performance. Thus the speech input indicating the action of pointing would expand the domain of attraction for the point gesture, thereby facilitating the recognition of that gesture. Similarly, the recognition of a particular hand gesture by the sensor can be used as a visual input to aid in speech interpretation.

⋆⋆⋆⋆ Design robust hardware for real time hand gesture recognition. The analog implementation of the hand motion sensor should demonstrate the feasibility of the approach. A long term goal would be to design analog circuitry which is compatible with VLSI implementation.

3.9 Equipment requirements

The research for the first year will primarily involve the analysis, design, and simulation of the proposed hand gesture recognition system. Existing equipment in the Communication Systems Research Laboratory will be used for this purpose. This equipment includes;

1. Sun SPARC station for analysis.
2. Silicon Graphics for simulations
3. Pipe processor and imaging equipment for generating inputs.

Initially, simulations of the dynamics will be used with either synthetic inputs or stored real world images. Later, when the analog hardware is designed the real time performance of the hand motion sensor can be evaluated.

The analog implementation of the sensor will begin in the second year of the project. Off-the-shelf equipment will be used when possible. Specialized equipment will have to be built in the lab or through contracts with external vendors. A partial list of equipment includes;

1. Sun computer to be used as a controller.
2. A/D (analog-to-digital) and D/A (digital-to-analog) board for the Sun.
3. Signal generator.
4. Oscilloscope.
5. Miscellaneous electronic components.

3.10 Contributions

In this section the main contributions to hand gesture recognition and teleconferencing are described.

Hand gesture recognition

Nonlinear systems have been investigated mathematically and used to model physical phenomena. Recently there has been considerable interest in using the unique properties of nonlinear systems to design specialized sensors. This project represents the first attempt to apply dynamical systems concepts to detect hand gestures. In the course of the project, new methods will be introduced for;

1. Designing a motion sensor based entirely upon analog concepts.
2. Detecting coordinated motion among the fingers.
3. Detecting partially occluded motion.

Teleconferencing

A cooperative work environment for teleconferencing should facilitate the use of multiple communication modalities. Such a system should impose as few constraints upon the user as possible. Toward this end, this project is expected to make the following contributions;

1. Real time recognition of hand gestures independent of the complexity of the hand motion.
2. Allow the use of more natural gestures.
3. Facilitate the joint use of language and vision in a cooperative work environment.

4 Application of Dynamics to Low-Level Sensing

The recognition of hand gestures by mapping the hand motion onto limit cycle attractors of a dynamical system introduces many new concepts. Familiarity with these concepts can most easily be gained by seeing their application to simple problems. The problems described in this section have been chosen to be as simple as possible to illustrate the basic concepts and still exhibit most of the rich behaviors of the dynamical systems which will be used for hand gesture recognition.

4.1 What is a dynamical system?

The information processing capabilities of a dynamical system depend upon the local behavior of attractors, or limit sets, in the state space of the system. The *model dynamics* in its most general form is a system of ordinary differential equations depending upon parameters λ such that

$$\dot{\mathbf{x}} = f_\lambda(\mathbf{x}) = f(\mathbf{x}, \lambda) \tag{3}$$

where $\mathbf{x} \in \mathcal{R}^n$, $\lambda \in \mathcal{R}^k$, and $f_\lambda : \mathcal{R}^n \to \mathcal{R}^n$ or $f : \mathcal{R}^n \times \mathcal{R}^k \to \mathcal{R}^n$. The system in Eq. (3) may be represented by a *vector field*, $\mathbf{x}_\lambda$. Solutions of this system are described by the *flow* $\mathbf{\Phi}_\lambda : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^n$ with $\Phi_\lambda(\mathbf{x}, t) = \mathbf{x}_\lambda(t)$ representing the value of $\mathbf{x}$ at time t and initial condition $\mathbf{x} = \mathbf{x}_\lambda(0)$. Individual curves $\mathbf{x}_\lambda : \mathcal{R} \to \mathcal{R}^n$ defined over the time interval Δt are *trajectories* of the flow $\Phi_\lambda(\mathbf{x}, t)$.

In general, the qualitative properties of the trajectories depend upon the parameters λ. The dynamical system is *structurally stable* if the qualitative properties of the trajectories remain unchanged for $\lambda \in \lambda_0 + \delta\lambda$ where $\delta \neq 0$. A *bifurcation* is said to occur at the parameter value λ_0 if the flows Φ_{λ_1} and Φ_{λ_2} are qualitatively different when both λ_1 and λ_2 are in the neighborhood of λ_0. In the analysis of bifurcation phenomena in nonlinear dynamical systems, we shall be interested in the local space or *manifold* on which the system acts. The behavior near fixed points and periodic orbits is described in terms of a reduced dimensional space called the *center manifold*, which is an invariant manifold tangent to the center eigenspace.

Bifurcation theory will be applied in this project to analyze various configurations of the dimensionless form of the double scroll system[12, 13, 14], also known as Chua's circuit

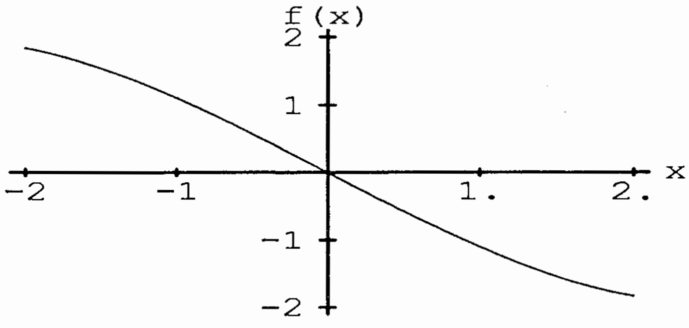


Figure 1: Cubic nonlinearity of the modified Chua's circuit.

$$\begin{aligned} \dot{x} &= \alpha\,[y - x - f(x)] \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned} \tag{4}$$

where

$$f(x) = m_1 x + \frac{1}{2}(m_0 - m_1)[|x+1| - |x-1|]. \tag{5}$$

In most previous analysis, the parameters are fixed with the values $m_0 = -\frac{8}{7}$ and $m_1 = -\frac{5}{7}$, and Eq. (4) is treated as a 2–parameter dynamical system with α and β as bifurcation parameters.

In this project we examine the case where $f(x)$ is a cubic function

$$f(x) = c_0 x + c_1 x^3. \tag{6}$$

in which a least squares fit of Eq. (6) to the piecewise-linear function Eq. (5) yields $c_0 = -\frac{7}{6}$ and $c_1 = \frac{1}{16}$. The shape of the cubic nonlinearity of $f(x)$ is illustrated in Fig. 1. The cubic nonlinearity in Chua's circuit gives rise to limit cycle oscillations for certain parameter ranges.

A limit cycle constitutes a particular trajectory in the xyz-coordinates of the dynamical system. The temporal responses of a pair of Gabor filters in phase quadrature constitute a trajectory in uv-coordinates. One objective of the normal form analysis is to construct a mapping between the uv-coordinates and the xyz-coordinates, so that Chua's circuit may be used as a convenient model for more complex visual processing based upon Gabor filters.

4.2 Normal form analysis

The equilibria of Eq. (4) are obtained from the condition $\dot{\mathbf{x}} = 0$ for $\mathbf{x} = (x, y, z)$. The three equilibria for this system are $p_0 = (0,0,0)$ and $p_{\pm 1} = (\pm\frac{2}{3}\sqrt{6}, 0.0, \mp\frac{2}{3}\sqrt{6})$. We shall be primarily interested in the behavior of the system near the stable equilibria at $p_{\pm 1}$. The equilibrium at the origin p_0 is unstable; therefore, there are no stable fixed points or limit cycles associated with this region of the state space.

The procedure for isolating the complex dynamics in the neighborhood of the equilibria at $p_{\pm 1}$ is based upon the method of Takens[20, 16]. Alternative methods for computing normal forms may be found in [17, 21, 22]. We give here an outline of the steps in this analysis.

I. The first step is to construct the real Jordan form for Eq. (4) through a linear change of coordinates. The Jordan form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \sigma & -\omega_0 & 0 \\ \omega_0 & \sigma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} f(u,v,w) \\ g(u,v,w) \\ h(u,v,w) \end{bmatrix} \tag{7}$$

has a linear part which is in block diagonal form plus a matrix containing higher order terms. The linear part of this system is characterized by a single pair of complex eigenvalues $\sigma \pm i\omega_0$ and a real eigenvalue γ.

II. The dimension of Eq. (7) is increased by one when the parameter σ is treated as an additional variable in the partial bifurcation analysis. The equivalent expanded system

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{\sigma} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} u \\ v \\ \sigma \\ w \end{bmatrix} + \begin{bmatrix} f_\sigma(u,v,w) \\ g_\sigma(u,v,w) \\ 0 \\ h_\sigma(u,v,w) \end{bmatrix} \tag{8}$$

now has dimension $n = 4$ with the condition $\dot{\sigma} = 0$. This expanded system has a pair of pure imaginary eigenvalues $\pm i\omega_0$, a zero eigenvalue, and a real eigenvalue γ.

III. The dimensionality of this system is reduced by a diffeomorphism (a smooth invertible map) which projects the generalized eigenspace with zero real part onto the center manifold. The dynamics near the center manifold are governed by

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \sigma \end{bmatrix} + \begin{bmatrix} f_c(u,v,\sigma) \\ g_c(u,v,\sigma) \\ 0 \end{bmatrix} \tag{9}$$

which has a pair of pure imaginary eigenvalues $\pm i\omega_0$ and a single zero eigenvalue.

IV. The dimensionality of this system can be further reduced by a normal form calculation yielding a nonlinear change of coordinates

$$\tilde{u} = u_c + \tilde{p}(u_c, v_c, \sigma) \qquad \tilde{v} = v_c + \tilde{q}(u_c, v_c, \sigma) \tag{10}$$

which transforms the dynamics on the center manifold into a normal form[16]. The procedure for deriving the normal form

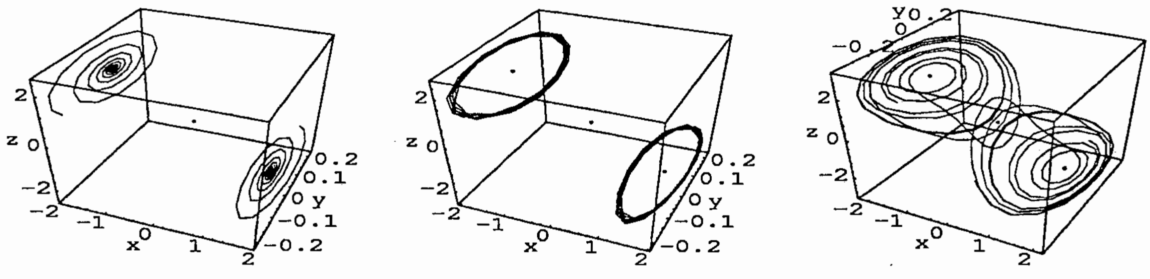
$$\begin{aligned} \dot{\tilde{u}} &= -\omega_0\tilde{v} + a_1\sigma\tilde{u} - a_2\sigma\tilde{v} + a_3(\tilde{u}^2+\tilde{v}^2)\tilde{u} - a_4(\tilde{u}^2+\tilde{v}^2)\tilde{v} + a_5\sigma^2\tilde{u} - a_6\sigma^2\tilde{v} \\ \dot{\tilde{v}} &= \omega_0\tilde{u} + a_2\sigma\tilde{u} - a_1\sigma\tilde{v} + a_4(\tilde{u}^2+\tilde{v}^2)\tilde{u} + a_3(\tilde{u}^2+\tilde{v}^2)\tilde{v} + a_6\sigma^2\tilde{u} + a_5\sigma^2\tilde{v} \\ \dot{\sigma} &= b_1(\tilde{u}^2+\tilde{v}^2) + b_2\sigma^2 + b_3(\tilde{u}^2+\tilde{v}^2)\sigma + b_4\sigma^3 \end{aligned} \tag{11}$$

for the dynamics on the center manifold of the scroll system is described in[4].

V. The bifurcation function

$$\dot{\rho} = a_1\sigma\rho + a_3\rho^3 + \mathcal{O}(|\rho^4|) \tag{12}$$

evaluated up to order 3 with $\rho = \sqrt{\tilde{u}^2 + \tilde{v}^2}$, is used to examine the existence of fixed points and limit cycles of the model dynamics for Chua's circuit. Variation in the parameter σ produces



(a) (b) (c)

Figure 2: Three qualitatively distinct behaviors of the scroll dynamics arising from the same initial state $(x, y, z) = \pm(-1.7, 0.2, 1.2)$, but different control parameters: (a) trajectories toward the fixed points ($\alpha = 4.5$, $\beta = 14.0$), (b) the trajectories on the limit cycle ($\alpha = 6.6$ $\beta = 14.0$, and (c) chaotic wandering between the two center manifolds ($\alpha = 8.8$, $\beta = 14.0$).

the three qualitatively distinct behaviors in Fig. 2 as illustrated by; a) trajectories converging toward the fixed points at p_{+1} and p_{-1}, b) trajectories along the limit cycles centered at p_{+1} and p_{-1}, and c) chaotic wandering of a single trajectory between the same two stable equilibria.

The local analysis of the dynamical system provides an analytic method for constructing the normal form of the dynamics. The calculation of the normal form systematically reduces the coupling among the low order nonlinear terms for the Taylor approximation in the neighborhood of the equilibria. Once this analysis has been performed, a bifurcation function can easily be constructed from the normal form equations. The bifurcation function can then be used to predict the qualitative changes in the dynamics near the equilibria of the original system as the parameters are allowed to vary.

The objective of the bifurcation analysis is to first transform the model dynamics to a normal form where the nonlinear coupling among the lower order state variables is reduced as much as possible. The qualitatively distinct behaviors of this reduced system are then analyzed in terms of the reduced set of parameters. The transformation theory outlined above is based upon successive diffeomorphisms, or smooth maps with smooth inverses, therefore the inverse transformations for the normal form can be used to reconstruct the same qualitatively distinct behaviors in the original model dynamics.

The primary contribution of normal form analysis is to provide an analytic method for reducing the nonlinear interdependence among low order terms; thereby facilitating the synthesis of more complex systems. This synthesis is accomplished by the coupling of simpler systems through their normal form representations, then transforming back to the form of the original dynamics.

4.3 Providing inputs to the system

The recognition of hand gestures is based upon the detection of a coherent pattern of motion for the hand and fingers in the image sequence. A dynamical system serves as a model for the spatial and temporal changes which occur during the formation of the hand gesture. In order to keep this illustration as simple as possible, consider the x, y, z-coordinates of the dynamical system as the spatial coordinates of a single finger tip. The motion of the finger tip is then

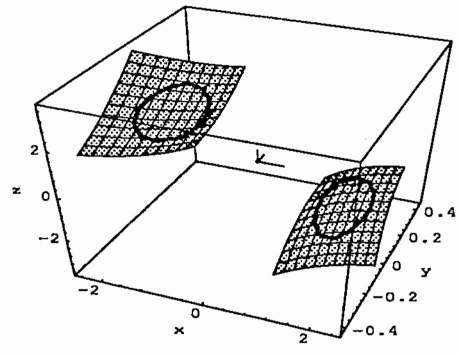


Figure 3: Limit cycle trajectories on the center manifold of Chua's circuit.

represented by $\{\dot{x}, \dot{y}, \dot{z}\}$. The motion of the finger is coherent in the sense that the components $\{x, y, z\}$ trace a smooth, continuous path in space. For example, the circular motion of a finger in the x, y-plane can be recognized by the condition that the x, y-coordinates of the model dynamics trace a circular path, and the z-coordinate is constant. Since the model dynamics can, in principle, assume any arbitrary form, it can be used to represent any arbitrary motion of the finger tip.

The coherency of the variables $\{x, y, z\}$ along the limit cycle of the model dynamics can be observed in Fig. 3. The limit cycle trajectory is constrained by the dynamics to lie on the surface of the center manifold, shown as the shaded region in Fig. 3. Due to the symmetry of the dynamics there are two center manifolds displaced from the origin.

A simple correspondence between the trajectories of the state variables of the model dynamics and the spatial/temporal motion of a single finger tip has been established. Given this representation, it is now possible to demonstrate the mechanism by which visual inputs can be provided to the model dynamics. Consider two dynamical systems: S_1 and S_2. Let S_1 be the dimensionless form of Chua's circuit

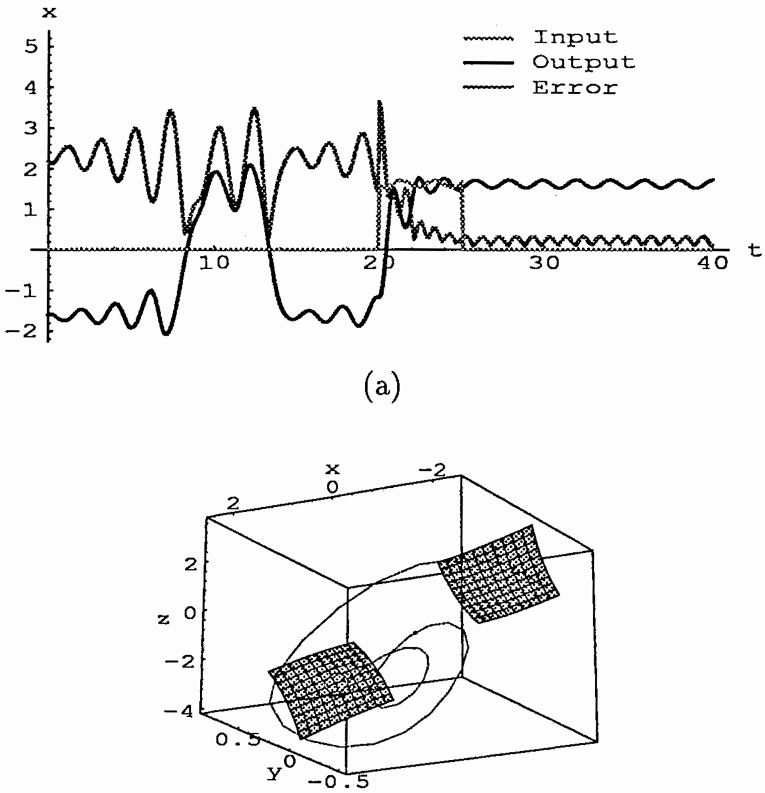
$$\begin{aligned} \dot{x}_1 &= \alpha_1 \left[y_1 - x_1 - f(x_1) \right] \\ \dot{y}_1 &= x_1 - y_1 + z_1 \\ \dot{z}_1 &= -\beta_1 y_1 \end{aligned} \tag{13}$$

and let S_2 be a modified version of S_1 given by

$$\begin{aligned} \dot{x}_2 &= \alpha_2 \left[y_2 - x_2 - f(x_2) \right] \\ \dot{y}_2 &= x_2 - y_2 + z_2 + \gamma(x_2 - x_1) \\ \dot{z}_2 &= -\beta_2 y_2 \end{aligned} \tag{14}$$

where the term $\gamma(x_2 - x_1)$ in Eq. (14) describes the coupling from S_1 to S_2 and the parameters α and β determine the qualitative properties of the system. Unless otherwise stated, $\beta_1 = \beta_2 = 14.0$ and $\alpha_1 \neq \alpha_2$. If $\gamma = 0$, then S_1 and S_2 are independent. When $\gamma \neq 0$, then the state of S_2 is influenced by the variable x_1. In particular, when $\gamma = 1$, then the contribution of x_2 in $\dot{y}_2$ is cancelled out and the behavior of $\dot{y}_2$ is determined by x_1, y_2 and z_2.

The dynamics of S_2 requires that it converge onto the center manifold, however, it is also constrained by the behavior of S_1 through the variable x_1. As a result of this additional



(a)

(b)

Figure 4: Synchronization of trajectories on the center manifold of Chua's circuit: (a) time series plot of the x-variable in the recognition system and the input signal, and (b) convergence of the recognition system onto the center manifold. (See text for details.)

constraint, S_2 converges onto the center manifold in such a way that x_2 becomes synchronized to x_1. Furthermore, the internal coupling among the variables x_2, y_2 and z_2 force y_2 and z_2 to synchronize to y_1 and z_1 respectively, even though these variables receive no direct input from S_1.

An example of the synchronization of S_2 from a chaotic state to the periodic limit cycle of S_1 is shown in Fig. 4. Initially, the detector system S_2 has the parameter $\alpha = 10.2$ set to produce a chaotic trajectory. From the time $t_1 = 20$ to $t_3 = 25$ an external input to $\dot{y}_2$ is provided from x_1 corresponding to a limit cycle oscillation in S_1 with $\alpha = 6.6$. Synchronization of phase occurs almost instantaneously as the chaotic system S_2 is drawn into the limit cycle, while the convergence onto the surface of the center manifold occurs exponentially. At $t_2 = 23$ synchronization has occurred so the parameters of S_2 are switch from the chaotic dynamics to $\alpha_2 = 6.6$ resulting in the sustained limit cycle oscillations. At $t_3 = 25$ the external input is removed and S_2 continues the limit cycle trajectory until the parameters are once again changed to begin processing the next input.

The central idea of this example is to show that the chaotic system can be trapped on the surface of the center manifold by an external input from a suitable signal. Although the motion to be detected corresponds to a trajectory on the surface of the center manifold, the chaotic state of the dynamical system is used initially as the motion detector since the chaotic system can move rapidly through its state space. Furthermore, the trapping phenomenon occurs very rapidly for the chaotic system and is independent of initial conditions for either the detector system S_2 or the source S_1. An additional property not illustrated in this example is that a small fragment of the limit cycle trajectory of the source system S_1 is generally sufficient to trap the detector system on the appropriate attractor. This latter property will be useful for detecting short fragments of occluded hand gestures.

Finger tip tracking

In the above example one dynamical system was used to provide the input for another similar system. In this case the three dimensional model dynamics S_2 received its input from only one variable, x_1, however synchronization occurred among all variables in the 3-D space of the model dynamics. The input to S_2 may come from a state variable, such as x_1, in another dynamical system, or the input may be generated as a result of visually tracking the position of a finger tip. The model dynamics will rapidly become synchronized (eg. the finger motion is "recognized" by the model dynamics) to the finger tip motion provided that the motion is consistent with the internal behavior of the model dynamics. In Fig. 4(a), the limit cycle oscillations of x_1 were consistent with the limit cycle oscillations in the model dynamics of S_2, even though the parameters of S_2 were set to produce chaotic trajectories.

In mathematical terms, synchronization occurs when the input signal lies on (or near) the surface of the center manifold. Thus, in Fig. 4(b), the motion of x_1 along the surface of the center manifold was used to force S_2 from a chaotic trajectory to a synchronous trajectory with S_1 on the center manifold. The linear part of the normal form dynamics for this case has the form

$$S_i = \begin{bmatrix} 0 & -\omega_i & 0 \\ \omega_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [\mathbf{X}] + G(\mathbf{X}) \tag{15}$$

in which the higher order nonlinear terms are contained in the function $G(\mathbf{X})$. The center manifold for this system was previously discussed in Fig. 3. Furthermore, new 3-D motion patterns can be detected by algebraically transforming the surface of the center manifold.

The hand gesture recognition paradigm

The schematic in Fig. 5 illustrates the proposed paradigm for hand gesture recognition using chaotic dynamical systems. The recognition system is initially in a chaotic state allowing rapid access to distant regions of its state space. The attractors associated with different finger trajectories exist in different regions of the state space. The presentation of an input trajectory of the finger tip rapidly drives the system toward the center manifold of the attractor which most closely resembles the input signal. A limit cycle oscillation on the surface of the center manifold was previously used in Fig. 4 to illustrate this property. The recognition system will remain synchronized to the input signal for as long as it is present.

One possible mode of operation is to use the external input to draw the recognition system close to the appropriate attractor, then switch from chaotic dynamics to limit cycle dynamics to allow the system to converge onto the center manifold of the local attractor. This would

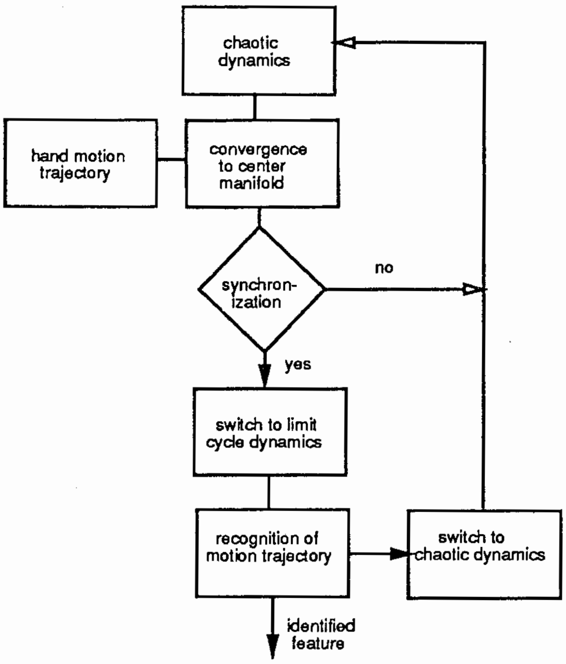


Figure 5: Control flow diagram illustrating the dynamical system's paradigm for hand gesture recognition.

allow the system to converge to a noise free state, and also allow it to continue to output the recognized motion even when the input is cut off due to occlusion. Once the recognition task is complete, the dynamics can be switched back to the chaotic regime and the detector is ready to classify the new input signal.

Hand gesture recognition

The motion of a single finger tip can be traced by a single dynamical system, S_i. The motion of the hand can be detected by multiple dynamical systems by using one system for each finger tip. The normal form of the hand recognition system, $\mathcal{S}$ is

$$\mathcal{S} = \begin{bmatrix} \begin{bmatrix} 0 & -\omega_1 & 0 \\ \omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & & H(\mathbf{x}) \\ & \begin{bmatrix} 0 & -\omega_2 & 0 \\ \omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \\ H(\mathbf{x}) & & \ddots \end{bmatrix} [\mathbf{X}] + G(\mathbf{X}) \qquad (16)$$

where each diagonal block contains a 3-D dynamical system S_i. The coupling among the S_i is represented by $H(\mathbf{X})$ and the higher order nonlinear terms are represented by $G(\mathbf{X})$. The recognition of a particular hand gesture requires that each diagonal block be designed to respond to the trajectory of one finger tip in the overall hand gesture. Just as each 3-D system had internal coupling to insure the coherency among all state variables, the composite system $\mathcal{S}$ must have internal coupling among the subsystems along the diagonal of the normal form matrix. The design of such internal coupling is an open research question at this time.

Multiple gestures from coupled systems

A dynamical system capable of recognizing a single hand gesture is not particularly useful, especially when it is necessary to identify various gestures for pointing, grasping, and releasing. It was previously shown that a single 3-D dynamical system was capable of tracking one motion of a single finger tip. The coupling of multiple 3-D systems within a composite system was subsequently proposed for recognizing a single hand gesture. The linear part of the normal form expression for the hand recognition system is

$$\mathcal{S} = \begin{bmatrix} A & & & & H(\mathbf{x}) \\ & A & & & \\ & & A & & \\ & & & A & \\ H(\mathbf{x}) & & & & A \end{bmatrix} [\mathbf{X}] + G(\mathbf{X}) \tag{17}$$

where each A_i is a 3-D dynamical system designed to detect a single motion pattern for a single finger.

In order to detect multiple trajectories with a single dynamical system it is necessary to use more complex dynamics. Multiple trajectories corresponding to different finger motions can be detected by increasing the dimensionality of the dynamical system. In this case the dimensionality of the diagonal blocks, A_i must be increased to realize the more complex dynamics. This results in the development of a hierarchical set of dynamical systems in which individual finger tip motions are detected by the diagonal blocks A_i and different hand gestures are detected by the composite system $\mathcal{S}$.

Advantage of chaotic dynamics

The parameters of the model dynamics during the motion recognition phase are chosen to produce chaotic trajectories. Since there are two equilibria, the chaotic trajectory produced by the dynamical system is continually shifting from the center manifold surface at one equilibria to the center manifold at the other equilibria. This has the effect of maintaining the system in a high energy state and allowing rapid transitions from one center manifold to the other. This means that the presentation of a coherent set of inputs (corresponding to a particular gesture) will cause the system to rapidly converge onto the center manifold whose surface corresponds most closely to the input signal. Rapid convergence of the dynamical system from arbitrary initial states (the initial state is random since the trajectory is chaotic prior to the input of the finger tip position) is a critical requirement for real time recognition of multiple hand gestures.

Linear systems are inadequate for hand gesture recognition because the convergence rate is typically quite slow. Furthermore, linear systems obey the principle of *superposition*, which means that the response is proportional to the sum of all inputs. Thus linear systems degrade in the presence of noise and are unable to discriminate between two conflicting inputs. Computer algorithms are inadequate for the hand gesture recognition task since invariant properties of the gesture must be discovered in order to make the recognition task computationally efficient. Such invariant properties are difficult to obtain and are *not* guaranteed to exist for all gestures.

Nonlinear systems with chaotic dynamics overcome these deficiencies. The nonlinearities allow rapid motion through the state space of the system, thus allowing rapid convergence. Additionally, nonlinear systems are not subject to superposition. Consequently, the discrimination between two input signals is achieved simply by the inherent properties of the system. Furthermore, invariants are not necessary since chaotic systems respond to any fragment of the input signal reguardless of its initial state.

5 Conclusion

The proposed approach for real time recognition of hand gestures has far ranging implications for constructing a cooperative work environment for teleconferencing. Human beings are highly skilled at integrating information from multiple sources and resolving ambiguities in a noisy environment. Dynamical systems exhibit a similar capacity to integrate multiple sources of information and should therefore help to provide a convenient interface between man and machine.

The analog properties of the dynamical systems guarantee real time performance for hand gesture recognition. The motion which occurs during the formation of the hand gesture provides critical inputs for the dynamical systems. This suggests that the use of motion to convey meaning in teleconferencing will acquire increasing importance.

The ability of dynamical systems to integrate information from multiple sources has important implications for the interaction between language and vision. The behavior of a dynamical system is determined by the parameters of that system. Bifurcation analysis shows that this behavior remains qualitatively the same over certain parameter ranges. Although the qualitative behavior remains the same, the detailed behavior may be modified by selected parameter changes. Thus a bias toward the recognition of certain hand gestures can be introduced by changing the system parameters. The detailed motions of the hand gestures to be recognized can also be taylored to individuals in the teleconferencing environment.

Speech input provides one possible source of external interactions with the proposed dynamical systems. The prediction of a "pointing" act from the dialog interpretation may be used to bias the dynamical system to recognize that particular gesture. Alternatively, the recognition of an individual's voice may be used to taylor the hand gesture recognition systems to detect the hand gestures particular to that individual. This would be accomplished by simply adjusting the parameters of the dynamical system. The generic properties of the system would remain the same, however the detailed behavior of the system would change for each individual.

Dynamical systems have unique properties which may facilitate the interaction between man and machine. This project begins to explore these properties by developing an analog sensor for hand gesture recognition.

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