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Abstract

The projective invariants used in computer vision today are permutation-sensitive since their value depends on the order in which the features were considered in the computation. We derive, using tools from representation theory, the projective and permutation (p^2) invariants of the four collinear and the five coplanar points configurations. The p^2-invariants are insensitive to both projective transformations and changes in the labeling of the points. When used as model database indexing functions in object recognition systems the p^2-invariants yield a significant speedup.

Keywords: projective invariants, permutation invariants, feature indexing, object recognition

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1 Introduction

Invariants of simple feature sets (e.g., configurations of a few points or lines) became a frequently used tool in computer vision. Given the set of features s and an operator g belonging to a transformation group, the invariant $I[\cdot]$ must satisfy $I[\mathrm{s}] = I[\mathrm{gs}]$ for any g in the group. If g is a projective transformation projective invariants are obtained. An extensive survey of the applications of projective invariants in computer vision can be found in [13].

Most projective invariants are permutation-sensitive. That is, the value of a projective invariant depends on the order in which the features were considered in its computation. A different ordering of the set, i.e., associating indices with the elements in a different way, usually yields a different value for the projective invariant. Interchanging the indices of the elements is equivalent with a permutation group acting on the set. The permutation group has its own permutation invariants, expressions whose value is unchanged by the reordering of the elements in the set.

Let the projective transformation $\mathbf{T}$ and the permutation π act on the set s. In this paper we derive invariants of s under both, the group of projective transformations and the group of permutations. A projective invariant $Q[\cdot]$ satisfies the condition $Q[\mathrm{s}] = Q[\mathrm{Ts}]$ for any (say) planar projective transformation. The permutation invariant $P[\cdot]$ satisfies the condition $P[\mathrm{s}] = P[\pi\mathrm{s}]$ for all the possible permutations of the indices labeling the features in the set. A projective and permutation (p^2) invariant $J[\cdot]$ of the set s, must satisfy $J[\mathrm{s}] = J[\pi\mathrm{s}] = J[\mathrm{Ts}] = J[\pi\mathrm{Ts}] = J[\mathrm{T}\pi\mathrm{s}]$ for any $\mathbf{T}$ in the transformation group and any π in the permutation group. The value of $J[\cdot]$ remains the same when the set undergoes a projective transformation. It is also independent of the way the indices are associated with the features in either the original or in the transformed set.

Invariants provide a fast indexing method into data-bases of models and are often used in object recognition systems (e.g. [8], [16], [18]). However, the employed invariants are permutation-sensitive and therefore an invariant has to be stored for any of the possible orderings of the set. When two sets are matched by similar values of a p^2-invariant the order of elements within the sets becomes irrelevant. The sets are matched independent of their ordering and storage of one p^2-invariant per set suffices. The p^2-invariants can be used to design fast and robust feature correspondence algorithms [12]. In this paper we give the expressions of the p^2-invariants of projectively transformed collinear and coplanar point configurations. The p^2-invariants are obtained exploiting the properties of the fundamental projective invariant, the cross-ratio.

Invariants of transformation groups were also used to derive optimal filters for pattern recognition tasks [9], [10]. The employed technique is based on representation theory. In representation theory the elements of a group are mapped into matrices. Group operations become matrix multiplications, and the invariants of the group are found by identifying the invariant subspaces in the corresponding vector space. The definition of the vector space and the mapping of group elements into matrices, are specific for the problem under consideration. For an introduction to the general theory of group representations see [14], and for a comprehensive treatment of finite groups see [3].

In this paper we use representation theory to find the p^2-invariants of the basic 1D and 2D point configurations. In Section 2 it is shown that four collinear points cannot have a nontrivial linear first-order p^2-invariant. We will however construct a function that is invariant under permutations up to a possible change of the sign. In Section 3 the second-order p^2-invariants of the four collinear points configuration are obtained. In Section 4 the 2D cross-ratios of five coplanar points are examined and the p^2-invariants of the configuration are derived. In Section 5 the sensitivity of p^2-invariants to noise is investigated through computer simulations. The importance of p^2-invariants for computer vision algorithms is discussed in Section 6.

2 Cross-Ratio in One Dimension

In projective spaces points and lines are dual. We use only point configurations, but the obtained results are valid for the dual line configurations as well.

The fundamental projective invariant of four points on a line or five points in a plane is the cross-ratio. The importance of cross-ratio for computer vision is well known, e.g. [4]; pp. 407-414. Other projective invariants, like the two invariants of two coplanar conics, can be defined in terms of cross-ratios [15]. The cross-ratio can be generalized for coplanar configurations in N-dimensional spaces [2].

We use the following procedure to derive the p^2-invariant of four collinear points:

1. Define the cross-ratio. The cross-ratio is a permutation-sensitive projective invariant.

2. Generate all the possible cross-ratio expressions for the permutation group acting on the indices of the four points. This defines an ensemble of six expressions.

3. Represent the relation between the point index permutations and the cross-ratio expressions by matrices operating on a vector space.

4. Find the invariant subspaces of the vector space, i.e., the subspaces which map into themselves under the action of the permutation group.

5. Substitute in the definition of a subspace the corresponding cross-ratio expressions. The result is a p^2-invariant of the four collinear point configuration.

The p^2-invariant of four collinear points, thus, is built in two steps. First, the influence of projective transformations is removed by using only cross-ratios. The effect of permutations is eliminated by finding the invariant associated with the ensemble of cross-ratio expressions. The p^2-invariants of five coplanar points will be derived using the results obtained for the four collinear points.

Let four distinct points on a line A_i, $i = 1 \dots 4$; have the coordinates x_i. The cross-ratio of the configuration is defined as

$$\lambda = \langle A_1 A_2 A_3 A_4 \rangle = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \,. \tag{1}$$

It can be shown (e.g. [17]; pp. 12-19) that the cross-ratio is invariant under one-dimensional projective transformations of the four point configuration. However, the cross-ratio is permutation-sensitive since a permutation π of the indices of the four points changes its value. Note that a point index permutation means a different ordering of the same four collinear points in the cross-ratio computation. The corresponding change in the cross-ratio is denoted by the mapping Π

$$\Pi(\lambda) = \left\langle A_{\pi(1)} A_{\pi(2)} A_{\pi(3)} A_{\pi(4)} \right\rangle \,. \tag{2}$$

where the $\pi(i)$ are nonequal and take values between 1 and 4. The ensemble of all point index permutations form the symmetrical group $\mathcal{S}_4$. Let a permutation in $\mathcal{S}_4$ be denoted as π_k. If π_{k_1} and π_{k_2} are two permutations in $\mathcal{S}_4$, their composition $\pi_{k_1}\pi_{k_2}$ is also a permutation in $\mathcal{S}_4$. The effect of the combination of two point index permutations on the cross-ratio is the combined mapping $\Pi_{k_1}(\Pi_{k_2}(\lambda))$.

Theorem 1 All the elements in the symmetric group $\mathcal{S}_4$ are finite products of three basic permutations

$$\begin{aligned} \pi_1 &: [1234] \mapsto [2134] \\ \pi_2 &: [1234] \mapsto [1324] \\ \pi_3 &: [1234] \mapsto [1243]\,. \end{aligned} \tag{3}$$

These basic permutations interchange only the k-th and the (k+1)-th point in equation (1). Thus, through a sequence of these basic permutations the four points can be rearranged into any order when computing the cross-ratio.

There are $4! = 24$ elements in the $\mathcal{S}_4$ group. However, they yield only six distinct cross-ratio values. This can be proved noticing that the three basic permutations π_k, defined in equation (3) generate only two distinct mappings Π_k for the cross-ratio λ:

$$\Pi_1(\lambda) = \frac{1}{\lambda} \qquad \Pi_2(\lambda) = 1 - \lambda \qquad \Pi_3(\lambda) = \frac{1}{\lambda} \, . \tag{4}$$

From Theorem 1 follows that all the possible mappings Π can be obtained by applying the operations $\frac{1}{q}$ and $1 - q$ to λ.

For example, the permutation $\pi = \pi_1 \pi_2$ yields the cross-ratio mapping:

$$\Pi(\lambda) = (\Pi_1 \Pi_2)(\lambda) = \Pi_1(\Pi_2(\lambda)) = \Pi_1(1 - \lambda) = \frac{1}{1 - \lambda} \, . \tag{5}$$

It can be easily shown that only six distinct mappings can be obtained this way. They yield the cross-ratio values:

$$\begin{array}{lll} \lambda_1 = \lambda & \lambda_2 = \frac{1}{\lambda} & \lambda_3 = \frac{1}{1-\lambda} \\ \lambda_4 = 1 - \lambda & \lambda_5 = \frac{\lambda - 1}{\lambda} & \lambda_6 = \frac{\lambda}{\lambda - 1} \, . \end{array} \tag{6}$$

The six λ_i expressions define a six-dimensional vector Λ. The functions in the vector Λ span a six-dimensional vector space $\mathbf{V}_1$.

The two mappings $\Pi_1(\lambda) = \Pi_3(\lambda)$ and $\Pi_2(\lambda)$ become the 6×6 matrices $\mathbf{P}_1 = \mathbf{P}_3$ and $\mathbf{P}_2$

$$\mathbf{P}_1 = \mathbf{P}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \, . \tag{7}$$

For example, multiplication of $\mathbf{P}_1$ with the six-dimensional column vector of λ_i describes the effect on the cross-ratio of interchanging the indices of the first or the second pair of points

$$\mathbf{P}_1\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 \\ \lambda_4 \\ \lambda_3 \\ \lambda_6 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} \Pi_1(\lambda_1) \\ \Pi_1(\lambda_2) \\ \Pi_1(\lambda_3) \\ \Pi_1(\lambda_4) \\ \Pi_1(\lambda_5) \\ \Pi_1(\lambda_6) \end{pmatrix} = \Pi_1(\Lambda)\ . \qquad (8)$$

From the definition of the λ_i-s in (6) follows immediately that (8) corresponds to substituting λ with $\frac{1}{\lambda}$ in the λ_i expressions.

Definition 1 A mapping Φ from $\mathcal{S}_4$ into the set of nonsingular $n \times n$ matrices that satisfies

$$\Phi(\pi_{k_1}\pi_{k_2}) = \Phi(\pi_{k_1})\Phi(\pi_{k_2}) \qquad (9)$$

for all $\pi_{k_1}, \pi_{k_2} \in \mathcal{S}_4$, is called an n-dimensional representation of $\mathcal{S}_4$.

In our case we have $\Phi(\pi_1) = \Phi(\pi_3) = \mathbf{P}_1$, $\Phi(\pi_2) = \mathbf{P}_2$, and the 24 matrices $\Phi(\pi)$ of the permutations $\pi \in \mathcal{S}_4$ are products of $\mathbf{P}_1$ and $\mathbf{P}_2$ (Theorem 1).

The subspace $\mathbf{U}$ of $\mathbf{V}_1$ is an *invariant subspace* if for all the elements $u \in \mathbf{U}$ and all the $\pi \in \mathcal{S}_4$ we have $\Phi(\pi)u \in \mathbf{U}$. An invariant subspace is *irreducible* if it contains no proper invariant subspaces. It can be shown using standard tools from representation theory (e.g., [5]; sections 2.1-2.3) that the space $\mathbf{V}_1$ can be decomposed into four irreducible subspaces having the dimensions: 1, 1, 2 and 2. The simplest invariant subspace is the one-dimensional subspace spanned by the vector $(1,1,1,1,1,1)$. In this subspace we have $\Phi(\pi)u = u$ for any $\pi \in \mathcal{S}_4$.

The irreducible subspaces contain vectors which are invariant under the mapping Φ of the permutation group $\mathcal{S}_4$. Projection of Λ on the first one-dimensional irreducible subspace provides the sought p^2-invariant. The projection is computed by taking the inner product between the vector $(1,1,1,1,1,1)$ and Λ:

$$I_1[\lambda] = \sum_{i=1}^{6} \lambda_i = 3 \qquad (10)$$

where the value 3 is obtained when the λ_i are substituted with their expression from (6). The invariant is trivial and of no interest in applications. We conclude that the linear combinations of the cross-ratio expressions cannot provide a nontrivial p^2-invariant under the mapping Φ of the permutation group $\mathcal{S}_4$ into a vector space. When the four points are permuted in the expression of the cross-ratio (1), first-order (linear) combinations of the resulting cross-ratio values do not yield nontrivial p^2-invariants.

Nontrivial invariants can, however, be obtained if we do not require full invariance but only invariance up to a multiplication with 1 or -1. This type of invariance is linked to the second one-dimensional subspace spanned by the vector $(1,-1,1,-1,1,-1)$. The vectors in this subspace transform as the alternating representation of $\mathcal{S}_4$ ([5]; section 2.3). This subspace yields the function

$$I_2[\lambda] = \frac{2\lambda^3 - 3\lambda^2 - 3\lambda + 2}{\lambda(\lambda-1)} \tag{11}$$

which transforms as the alternating representation of $\mathcal{S}_4$:

$$I_2[\lambda] = -I_2\left[\frac{1}{\lambda}\right] = I_2\left[\frac{1}{1-\lambda}\right] = -I_2[1-\lambda] = I_2\left[\frac{\lambda-1}{\lambda}\right] = -I_2\left[\frac{\lambda}{\lambda-1}\right] . \tag{12}$$

In applications the information contained in the sign of $I_2[\lambda]$ is often of no importance and the invariant $|I_2[\lambda]|$ can be used. In Figure 1 the graphs of the functions $-I_2[\lambda]$ and $|I_2[\lambda]|$ are given.

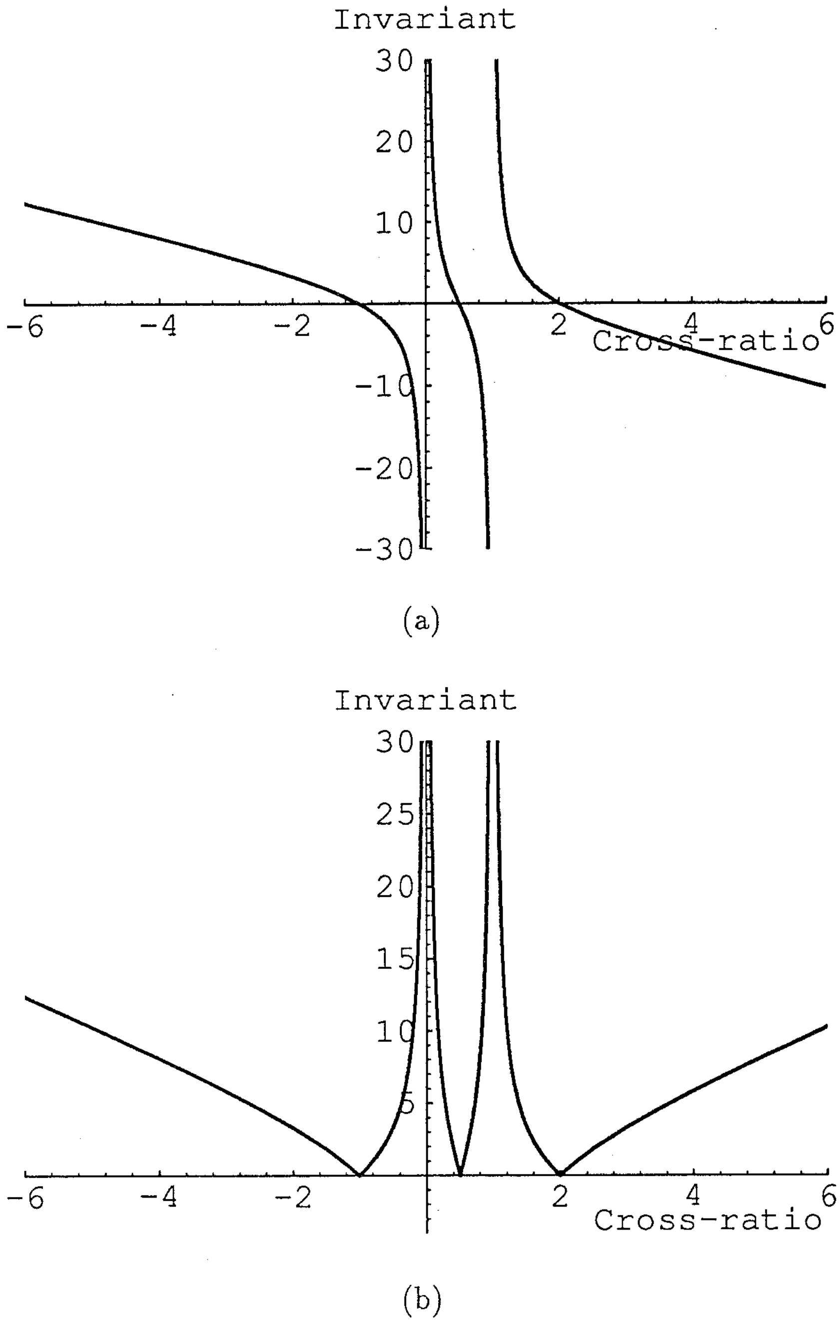
The study of the irreducible subspaces of dimension two leads to the functions

$$I_3[\lambda] = \frac{\lambda^2 - \lambda + 1}{\lambda(\lambda-1)} \qquad I_4[\lambda] = \frac{\lambda^2 - \lambda + 1}{\lambda - 1} . \tag{13}$$

which transform into linear combinations of each other. For example:

$$\begin{aligned} I_3\left[\tfrac{\lambda-1}{\lambda}\right] &= -I_4[\lambda] & \qquad I_3\left[\tfrac{\lambda}{\lambda-1}\right] &= I_4[\lambda] - I_3[\lambda] \\ I_4\left[\tfrac{1}{\lambda}\right] &= -I_3[\lambda] & \qquad I_4[1-\lambda] &= I_3[\lambda] - I_4[\lambda] . \end{aligned} \tag{14}$$

These complicated transformation rules make them uninteresting in practical applications.



(a)

(b)

Figure 1: (a) The invariant $-I_2[\lambda]$. (b) Its absolute value.

In the next section we show that with a different representation of $\mathcal{S}_4$, based on the pairwise products $\lambda_i\lambda_j$, useful p^2-invariants for the configuration of four collinear points can be found.

3 Second-Order P^2-Invariants

We now define a new mapping from $\mathcal{S}_4$ to matrices acting on a vector space $\mathbf{V}_2$. Of the 21 products $\lambda_i\lambda_j$, $i \geq j$; $i,j = 1 \ldots 5$, three are equal to 1. The remaining 18 products span an 18 dimensional vector space $\mathbf{V}_2$. Following the procedure described in the previous section we obtain an 18 dimensional representation of $\mathcal{S}_4$.

The space $\mathbf{V}_2$ can be decomposed into six one-dimensional and six two-dimensional irreducible subspaces. Two of the one-dimensional subspaces are connected to the alternating representation of $\mathcal{S}_4$ and lead to the invariant $I_2[\lambda]$ (11). Projection of the 18 dimensional vector of $\mathbf{V}_2$ into the other four one-dimensional irreducible subspaces yields the following four p^2-invariants:

$$
\begin{aligned}
J_1[\lambda] &= \frac{\lambda^6 - 3\lambda^5 + 3\lambda^4 - \lambda^3 + 3\lambda^2 - 3\lambda + 1}{\lambda^2(\lambda-1)^2} \\
J_2[\lambda] &= \frac{2\lambda^6 - 6\lambda^5 + 9\lambda^4 - 8\lambda^3 + 9\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda-1)^2} \\
J_3[\lambda] &= 3 \\
J_4[\lambda] &= -3.
\end{aligned}
\tag{15}
$$

These p^2-invariants can be regarded as being of second-order since they were generated using products of the cross-ratio expressions. Only the first two are nontrivial.

It can be shown that all linear second-order p^2-invariants of four collinear points are linear combinations of the four invariants (15). Some of these linear combinations are already known.

$$J_{11}[\lambda] = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2} = J_1[\lambda] + J_3[\lambda] \ . \tag{16}$$

The p²-invariant $J_{11}[\lambda]$ is often mentioned in the mathematical literature (e.g. [7], p.317; [1], p.127), and was used in computer vision by Maybank [11] to investigate nonplanar conics. The linear combination

$$J_{12}[\lambda] = \sum_{i=1}^{6} \sum_{j=1, j\neq i}^{6} \lambda_i \lambda_j = J_1[\lambda] + J_4[\lambda] \ . \tag{17}$$

is an intuitive p²-invariant which can also be derived as a symmetric function of the six λ_i [12]. Another symmetric function is

$$J_{13}[\lambda] = \sum_{i=1}^{6} \lambda_i^2 = -J_1[\lambda] + J_2[\lambda] - 0.5 \cdot J_3[\lambda] \ . \tag{18}$$

In Figure 2 the graphs of $J_1[\lambda]$ and $J_2[\lambda]$ are shown. The discontinuities at $\lambda = 0$ and $\lambda = 1$ correspond to violations of the four distinct points assumption.

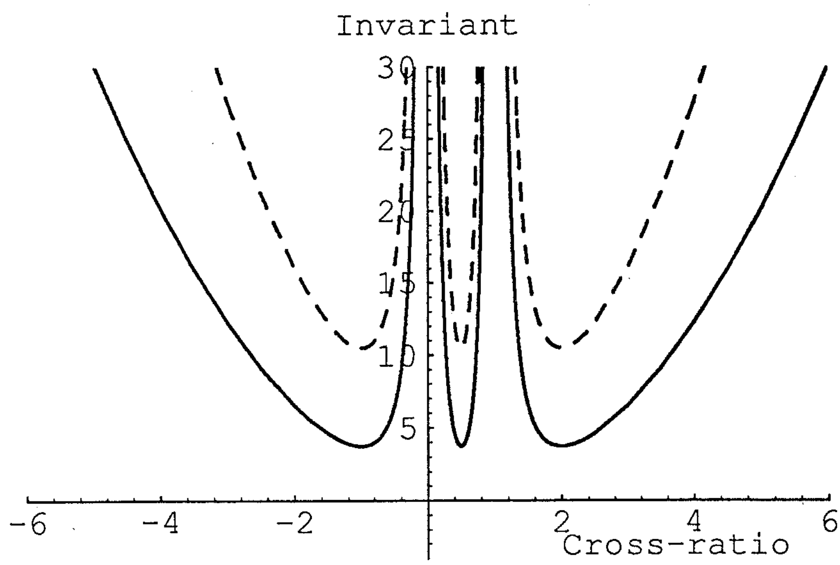


Figure 2: The invariants $J_1\,[\lambda]$ (solid line) and $J_2\,[\lambda]$ (dashed line).

Bounded p^2-invariants of the four points can be constructed using ratios of $J_1[\lambda]$ and $J_2[\lambda]$ since none of these functions has real roots. Another bounded p^2-invariant is the ratio of $J_{11}[\lambda]$ and $J_{12}[\lambda]$. In Figure 3 the dependence of

$$J_{14}[\lambda] = \frac{J_2[\lambda]}{J_1[\lambda]} \tag{19}$$

on λ is shown.

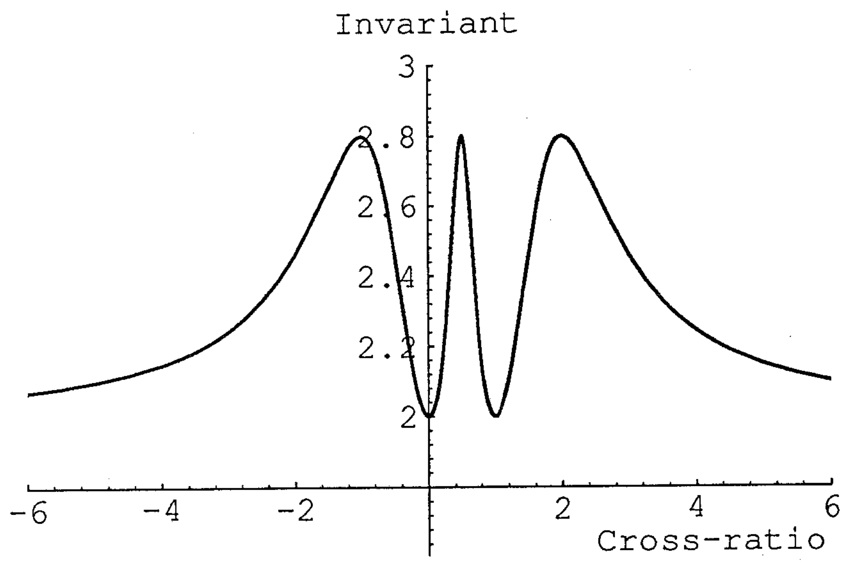


Figure 3: The bounded invariant $J_{14}[\lambda]$.

A comparison of Figures 1, 2 and 3 shows that all the invariants have the same structure consisting of six monotonic branches in the six intervals $(-\infty, -1]$, $(-1, 0)$, $[0, 0.5]$, $(0.5, 1]$, $(1, 2)$, $[2, \infty)$ covering the real axis. Their extrema are always at the boundaries of these intervals.

Let $J[\lambda]$ be any nontrivial p^2-invariant. If the equation $J[\lambda] = C$ has six simple solutions, then the roots correspond to the six cross-ratio values in (6). Only when the four points form a harmonic range are three multiple roots obtained. The three cross-ratio values are -1, 0.5, 2. Making use of a p^2-invariant, the configuration of four collinear points can always be associated with a cross-ratio in the interval [0, 0.5] without actually permuting the point indices.

The p^2-invariants for the four collinear points configuration were derived using the special structure of the resulting six cross-ratio expressions. In the two-dimensional case such structure cannot be found and the p^2-invariants must be derived by recognizing the underlying one-dimensional processes.

4 Cross-Ratio in Two Dimensions

In two dimensions a configuration of five points A_i, $i = 1 \ldots 5$; is required to define cross-ratios. In homogeneous coordinates the points are described by the triplets (x_i, y_i, z_i) and any nonsingular 3×3 matrix defines a planar projective transformation of the configuration. The points must be in arbitrary positions, i.e., no three collinear. Several equivalent definitions exist for the cross-ratio in two dimensions. Using the ratios of the areas of four triangles delineated by triplets of points is the computationally most convenient definition. One point is shared by all the four triangles. If this common point is A_1 then the cross-ratio is defined as

$$\mu = \frac{(\Delta A_1 A_2 A_4)(\Delta A_1 A_3 A_5)}{(\Delta A_1 A_3 A_4)(\Delta A_1 A_2 A_5)} = \frac{\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix} \begin{vmatrix} x_1 & x_3 & x_5 \\ y_1 & y_3 & y_5 \\ z_1 & z_3 & z_5 \end{vmatrix}}{\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{vmatrix} \begin{vmatrix} x_1 & y_2 & x_5 \\ y_1 & y_2 & y_5 \\ z_1 & z_2 & z_5 \end{vmatrix}} \tag{20}$$

where $(\Delta A_i A_j A_k)$ is the oriented area of the triangle defined by the points A_i, A_j and A_k. If the point A_i is a Cartesian point then we can choose $z_i = 1$, if it is an ideal point then z_i is zero. The cross-ratio is an absolute invariant of the group of planar projective transformations.

Using the different points as common point for the triangles, five different two-dimensional cross-ratios can be computed for a configuration of five coplanar points. Expressions of the other four cross ratios can derived from (20) by interchanging the two indices 1 and k for $k = 2, ..., 5$. Thus the cross-ratio associated with the point A_2 is:

$$\nu = \frac{(\Delta A_2 A_1 A_4)(\Delta A_2 A_3 A_5)}{(\Delta A_2 A_3 A_4)(\Delta A_2 A_1 A_5)} \,. \tag{21}$$

The cross-ratios μ and ν defined in (20) and (21) correspond thus to the configurations $\langle A_1 A_2 A_3 A_4 A_5 \rangle$ and $\langle A_2 A_1 A_3 A_4 A_5 \rangle$. Note that the usual definition of the cross-ratio (23) would involve cyclically increasing all the indices by one, i.e., the point configuration $\langle A_2 A_3 A_4 A_5 A_1 \rangle$ would be used. As will be shown below, these two definitions are related and our choice leads to simpler invariant expressions. Of the five cross-ratios only two are independent since the five points span a ten-dimensional space while a planar projective transformations has eight degrees of freedom. Any two of the five cross-ratios can be taken as the independent pair and the other three expressed function of them.

The two-dimensional cross-ratio is permutation-sensitive. The permutation-sensitivity can be analyzed using an equivalent definition in which the underlying one-dimensional processes can be recognized. The common point of the four triangles, say the point A_1, is taken as the center of perspective from which the remaining four points are projected on a line defined by two of them, say A_2 and A_3 (Figure 4a). The one-dimensional cross-ratio of the four collinear points $\langle B_5 A_2 A_3 B_4 \rangle$ is connected to the two-dimensional cross-ratio defined as in (20). The equivalence is immediate if we recognize that all the four triangles have the same altitude. When a different pair of points, say A_2 and A_4, is used to define the line, the new point range B_3, A_4, B_5, A_2 (Figure 4b) is in perspective correspondence with the range B_5, A_2, A_3, B_4 (Figure 4a). However, the order of the corresponding points changes and this may yield a different cross-ratio value, as was shown in Section 2. Haralick [6] used the one-dimensional definition to investigate the relations among the five two-dimensional cross-ratios

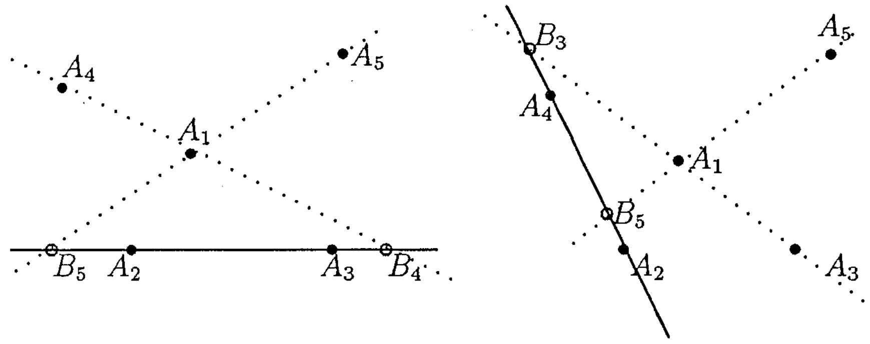


Figure 4: Two examples for defining the cross-ratio associated with the point A_1.

computed for the five points. To solve the issue of permutation-sensitivity he had to discriminate between a star and a tower topology for the configuration. The p^2-invariants to be derived below are insensitive to permutations of the five point indices and therefore independent of the configuration topology.

A planar projective transformation is uniquely defined when the correspondence of four points is known. Since the cross-ratio is a projective invariant we can take the points A_1 to A_4 as having the homogeneous coordinates

$$A_1 : (0,0,1) \qquad A_2 : (1,0,1) \qquad A_3 : (0,1,1) \qquad A_4 : (1,1,0) \ . \tag{22}$$

These four points define the projective coordinate system of the plane. The fifth point has the coordinates $A_5 : (x, y, 1)$. The two independent cross-ratios, computed from the configurations $\langle A_1 A_2 A_3 A_4 A_5 \rangle$ and $\langle A_2 A_1 A_3 A_4 A_5 \rangle$ are denoted as μ and ν.

When a given point is the center of perspective, there are $4! = 24$ possible permutations for the remaining four points. However, as the example in Figure 4 shows, these permutations generate projectively correspondent point ranges. The changes in the cross-ratio value are caused by changes in the order in which the points in the range are considered.

Let μ be the two-dimensional cross-ratio of a given point configuration. As in the one-dimensional case denote by π_k the permutation that interchanges the k-th and the (k+1)-th point and by $\Pi_k(\mu)$ the corresponding transformation of the two-dimensional cross-ratio (20).

Theorem 2 The basic permutations π_2, π_3 and π_4 yield two different transformations for the two-dimensional cross-ratios:

$$\Pi_2(\mu) = \Pi_4(\mu) = \tfrac{1}{\mu} \quad \text{and} \quad \Pi_3(\mu) = 1 - \mu \tag{23}$$

This theorem can be proved by an easy calculation.

An arbitrary permutation of five points can be written as the product of two permutations $\pi\pi^{(k)}$ where π is a permutation of the last four points and $\pi^{(k)}$ is the permutation that exchanges the first and the k-th point. Thus $\pi^{(k)} \neq \pi_k$ if $k \neq 1$. The corresponding cross-ratio transformations are $\Pi(\mu)$ and $\Pi^{(k)}(\mu)$ and thus the cross-ratio transformation of an arbitrary permutation can be written as $\Pi\left(\Pi^{(k)}(\mu)\right)$. Permutations of the last four points can always be expressed as products of π_2, π_3 and π_4. From Theorem 2 we obtain that the effect of Π on μ is one of the six basic transformations of the one-dimensional cross-ratio (6). The functions $J[\cdot]$ investigated in Section 3 are invariant under these transformations and we have:

$$J\left[(\Pi\Pi^{(k)})(\mu)\right] = J\left[\Pi(\Pi^{(k)}(\mu))\right] = J\left[\Pi^{(k)}(\mu)\right] = J^{(k)}\ . \tag{24}$$

The relation (24) shows that when the value of the two-dimensional cross-ratio is substituted into the expression of a one-dimensional invariant $J[\cdot]$ the obtained function depends only on the point used as the projection center. Permutations of the remaining four points have no influence on its value.

We can now proceed to our main result for the five coplanar points configuration.

Theorem 3 A permutation of the five points in the two-dimensional cross-ratio computation leads to the same permutation of the values $J^{(1)}, ..., J^{(5)}$.

If G is a group, H is a subgroup of G and $g \in G$ then the *right coset* Hg is defined as the set $\{hg : h \in H\}$. Two elements g_1 and g_2 in G are defined as equivalent if they are in the same coset. This defines an equivalence relation on G. This equivalence relation partitions G into disjunct cosets having the same number of elements.

We now compute the cosets for $G = \mathcal{S}_5$, the permutation group of five elements with H being the subgroup of all permutations of the last four elements. If the permutations in $\mathcal{S}_5$ are described by $[i_1 i_2 i_3 i_4 i_5]$ then the elements in H are of the form $[1 i_2 i_3 i_4 i_5]$. H has $4! = 24$ and $\mathcal{S}_5$ consists of $5! = 120$ elements. This gives five different right cosets of H in $\mathcal{S}_5$. The different cosets are: $H\pi^{(k)} = \{[k i_2 i_3 i_4 i_5] : i_l \neq k\}$ for $k = 1, ..., 5$. The J-functions are constant on a coset since $J[\Pi(\mu)] = J[\mu]$ for all $\pi \in H$ (Theorem 2).

Let μ and $\tilde{\mu}$ be the cross-ratios computed from the point configurations $\langle A_1 A_2 A_3 A_4 A_5 \rangle$ and $\langle A_{i_1} A_{i_2} A_{i_3} A_{i_4} A_{i_5} \rangle$ respectively. Then the values of the invariants are related by:

$$J^{(k)}[\tilde{\mu}] = J^{(k)}\left[\tilde{\Pi}(\mu)\right] = J\left[\Pi^{(k)}\left(\tilde{\Pi}(\mu)\right)\right] = J\left[\Pi^{(i_k)}(\mu)\right] = J^{(i_k)} \tag{25}$$

since the first element of the permutation $\pi^{(k)}\tilde{\pi}$ is i_k, as can be seen from the definition of $\pi^{(k)}$ and $\tilde{\pi}$.

The proof shows that the 120 different permutations in $\mathcal{S}_5$ yield exactly five different values of the invariant, one for each coset of H. These five values can be computed by selecting one permutation in each of the cosets, computing the corresponding cross-ratio and applying the J-function. There are only two independent two-dimensional cross-ratios. This implies that there are only two independent functions among $J^{(1)}, ..., J^{(5)}$. The relations among these functions are obtained by calculating the cross-ratios of the point configurations $\pi^{(k)}\langle A_1 A_2 A_3 A_4 A_5 \rangle$.

The corresponding cross-ratios are found by expressing $\Pi^{(k)}(\mu)$ as functions of the special cross-ratios μ and ν defined in (20) and (21).

Theorem 4 If μ and ν are the cross-ratios computed from the original point configuration and the configuration in which the points A_1 and A_2 were interchanged then the five J-values are:

$$\begin{array}{lll} J^{(1)} = J[\mu] & J^{(2)} = J[\nu] & J^{(3)} = J\left[\frac{\mu}{\nu}\right] \\ J^{(4)} = J\left[\frac{\nu-\mu}{\nu-1}\right] & J^{(5)} = J\left[\frac{\mu(\nu-1)}{\nu-\mu}\right] & \end{array} \quad (26)$$

Since the functions $J^{(k)}$ are invariant under permutations of the last four points in the configuration we can choose more symmetrical expressions as arguments for $J^{(4)}$ and $J^{(5)}$:

$$J^{(4)} = J\left[\frac{\nu-1}{\mu-1}\right] \quad J^{(5)} = J\left[\frac{\mu(\nu-1)}{\nu(\mu-1)}\right] . \quad (27)$$

Theorem 3 assures that the relations (26) and (27) remain valid when μ and ν are computed using points other than A_1 and A_2. Note, however, that ν must be computed by interchanging the points similar to (21). For each one-dimensional invariant$J[\cdot]$ we have constructed an ensemble of five functions for the configuration of five coplanar points. The five $J^{(k)}$ values can be combined in several ways to define p^2-invariants of practical use. When their five-dimensional vector is used for indexing, Theorem 3 assures that the point correspondence between the matched configurations is also recovered. In other applications an one-dimensional indexing function

$$K[\mu,\nu] = J[\mu] + J[\nu] + J\left[\frac{\mu}{\nu}\right] + J\left[\frac{\nu-1}{\mu-1}\right] + J\left[\frac{\mu(\nu-1)}{\nu(\mu-1)}\right]. \quad (28)$$

might be desired.

5 Noise Sensitivity

In the previous sections we have derived three different types of p^2-invariants for the four collinear points configuration: $I_2[\lambda]$ (11) connected to the alternating representation of the permutation group $\mathcal{S}_4$; linear combinations of the invariants $J_1[\lambda]$ and $J_2[\lambda]$ (15); and ratios of these linear combinations, e.g., $J_{14}[\lambda]$ (19). Any of these functions can be used to construct p^2-invariants for the five coplanar points configuration: either as a five-dimensional vector with the components $J^{(1)}, J^{(2)}, J^{(3)}, J^{(4)}, J^{(5)}$ (26, 27), or as a scalar $K[\mu, \nu]$ (28). The invariants $I_2[\lambda], J_1[\lambda]$ and $J_2[\lambda]$ have singularities at λ equal zero and one. While cross-ratio values of zero and one correspond to coinciding points on the line, in practice values close to these singularities can often be obtained. The invariant $J_{14}[\lambda]$, on the other hand, remains always bounded. We performed simulations to qualitatively investigate the noise sensitivity of the p^2-invariants for five coplanar points. The functions $J_{11}[\lambda]$ (16) and $J_{14}[\lambda]$ (19) were used.

In the first experiment we tested how reliably the point correspondence can be recovered from two matched configurations. Since a projective transformation modifies the metrical properties and thus changes the effect of noise, no such transformation was performed. We generated 1000 different point configurations with uniformly distributed integer coordinates in the range 0 to 1024. For each configuration the vector $J = (J^{(1)}, J^{(2)}, J^{(3)}, J^{(4)}, J^{(5)})$ was computed. Then the coordinates of the points were disturbed by adding uniformly distributed, integer valued noise. Twenty noise levels were employed. The order of the points was also changed by applying a randomly selected permutation. From the noisy, permuted configuration a new vector J' was computed.

In the absence of the noise, Theorem 3 assures that J' is a permuted version of J. The value of $J^{(k)}$ changes when noise is corrupting the point coordinates. The point correspondence was therefore recovered by computing the permutation π

such that J and $\pi(J')$ are ordered in the same way. The decision error was measured by the number of incorrect point correspondences averaged over the 1000 configurations. In Figure 5 the dependence of the decision error on the amount of added noise is shown. Both the unbounded $J_{11}[\lambda]$ and the bounded $J_{14}[\lambda]$ functions show similar behavior. This could be expected since both functions have the same six-branch structure and the allowed range of the $J^{(k)}$ values was not limited upward.

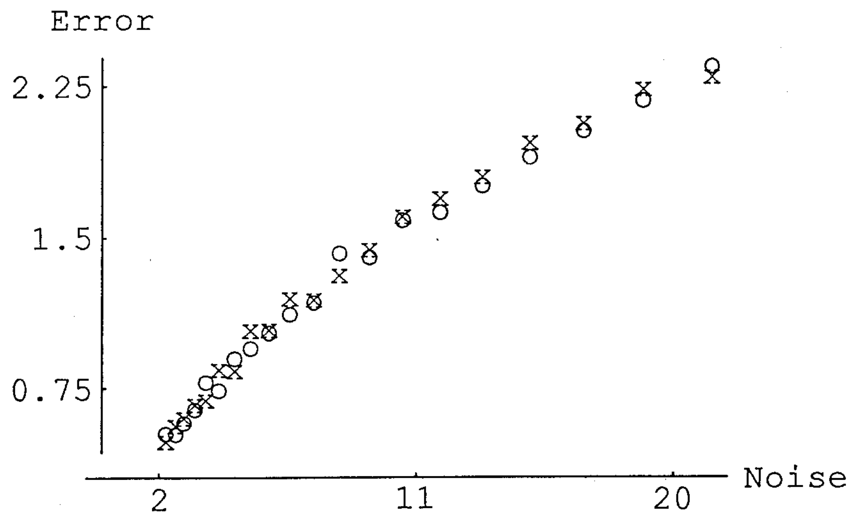


Figure 5: Average point correspondence error. $J_{11}[\lambda]$: o; $J_{14}[\lambda]$: x.

In practical situations, however, it is not recommended to have very large $J^{(k)}$ values, and the bounded function $J_{14}[\lambda]$ should be preferred in constructing the p^2-invariants. This can be seen by investigating the behavior of $K[\mu, \nu]$. The value of the invariant was computed for the original and its noisy, permuted counterpart. The mean squared difference of these two values computed from the 1000 configuration pairs was used as a measure of the noise sensitivity of the invariant. Note that we did not select the matching pairs out of the 1000 configuration pairs, which is a much stronger test for noise sensitivity. When the bounded function $J_{14}[\lambda]$ is used the mean squared difference increases monotonically with the noise

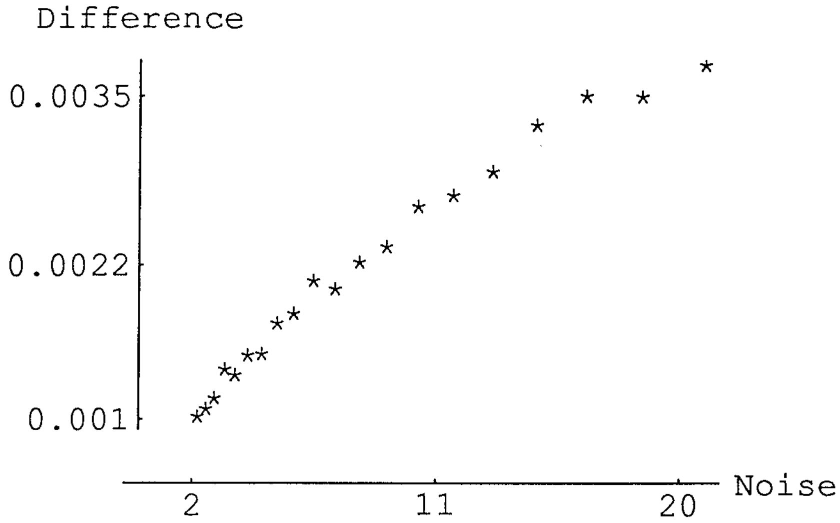


Figure 6: Average mean square difference of $K[\mu,\nu]$ computed with $J_{14}[\lambda]$.

level (Figure 6).

However, the differences obtained when $J_{11}[\lambda]$ is used can be arbitrarily large since often the cross-ratio is close to the singularities. These large values make any simple difference based matching procedure unreliable. Imposing an upper bound on the $K[\mu,\nu]$ values accepted in the matching procedure eliminates a significant percentage of the configurations as it is shown in Table 1.

Threshold	10^3	10^4	10^5
Rejection Rate	42%	18%	9.25%

Table 1: Percentage of configurations exceeding the upper bound for $K[\mu,\nu]$ computed with $J_{11}[\lambda]$.

We conclude from these simple noise sensitivity studies that in applications the p^2-invariants should be built with bounded functions $J[\cdot]$.

6 Discussion

The two-dimensional p^2-invariants proposed in this paper are of importance for developing fast and robust object recognition algorithms. In the algorithms used today, the invariant description of the model and its instances in the image depend on the order in which the features are used in computing the invariants [8], [16], [18]. The p^2-invariants do not depend on the selection order and a configuration of five coplanar points has always the same p^2-invariant independent of the way the indices are associated with the points.

The five $J^{(k)}$ functions can be combined into several p^2-invariants of the five coplanar points configuration. In an explicit approach the five-dimensional vector of the *ordered* set of the five values is used, i.e., the largest value is always taken as the first component. From Theorem 3 we have that this vector remains the same under any change in the indexing of the five points before and after a projective transformation. Five-tuples of points can therefore be matched by matching the vectors associated with them.

The matching procedure is optimal since it minimizes:

- the average time of finding a match, by being independent of the way the points are indexed;
- the average time of detecting a mismatch, by comparing the vector components sequentially.

In an implicit approach, the five $J^{(k)}$ functions define the one-dimensional p^2-invariant (28) for indexing. After matching two configurations uncertainty remains, however, about the precise point correspondence. Collapsing the ten degrees of freedom of the five points into a scalar value will significantly increase the probability of incorrectly matched configurations. The condition for robust performance and recovery of the point correspondence under such conditions is given in [12]. The noise sensitivity of both is currently under investigation.

To obtain reliable performance in the recognition of partial instances of multiple objects in an image (e.g., occlusions), the extracted features should be first grouped into topologically invariant units [16], [18]. The p^2-invariants proposed in this paper are associated with five-tuples of coplanar points. Their use as matching tools in object recognition algorithms, thus facilitates decomposition of a complex model into simple parts. Note also that two point configurations are matched without using information about the model to which the configuration may belong. Higher level information involving the invariant description of the entire model can then be used as an *independent* verification of the obtained result, or to recognize occluded objects.

We have described a systematic procedure to obtain point configuration invariants under simultaneous projective and permutation transformations of the points. The results are also valid for five arbitrary lines in a plane. Our methodology, based on the representation theory of finite groups, may become equally useful for developing transformation/permutation invariants for other configurations or transformation groups of importance in computer vision.

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