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Random Closed Sets: theory and applications

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1991.12. 2
（1991.11.29 受付）

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Random Closed Sets: theory and applications

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This report contains all the materials (slides, pictures) presented at a series of talks on the RACS theory, held at ATR Auditory and Visual Perception Research Laboratories in November 1991.

I. What is a Random Closed Set?
確率閉集合とは？

II. Links with other theories of Uncertainty
従来のあいまいさ理論との関係

III. Implementational issues and applications
実用にあたっての問題点

I. What is a Random Closed Set?
確率閉集合とは？

1. **review of basic topology (位相幾何学)**
   1.1 topological spaces (位相空間)
   1.2 open and closed sets (開集合、閉集合)
   1.3 continuity, semi-continuity
   (連続写像、半連続変数)
   1.4 compact sets (コンパクト集合)

2. **review of basic probability measure theory (確率測度論)**
   2.1 σ-algebras, events (加法族、事象)
   2.2 measures, probability measures
   (測度、確率測度)
   2.3 measurable mappings, random variables
   (可測写像、確率変数)
   2.4 Borel σ-algebras generated by a topology
   (ボレル加法族)

3. **Random Closed Sets (確率閉集合)**
   3.1 Hit or Miss topology (Hit/Miss位相)
   3.2 formal definition (定義)
   3.3 Choquet's theorem (Choquetの定理)

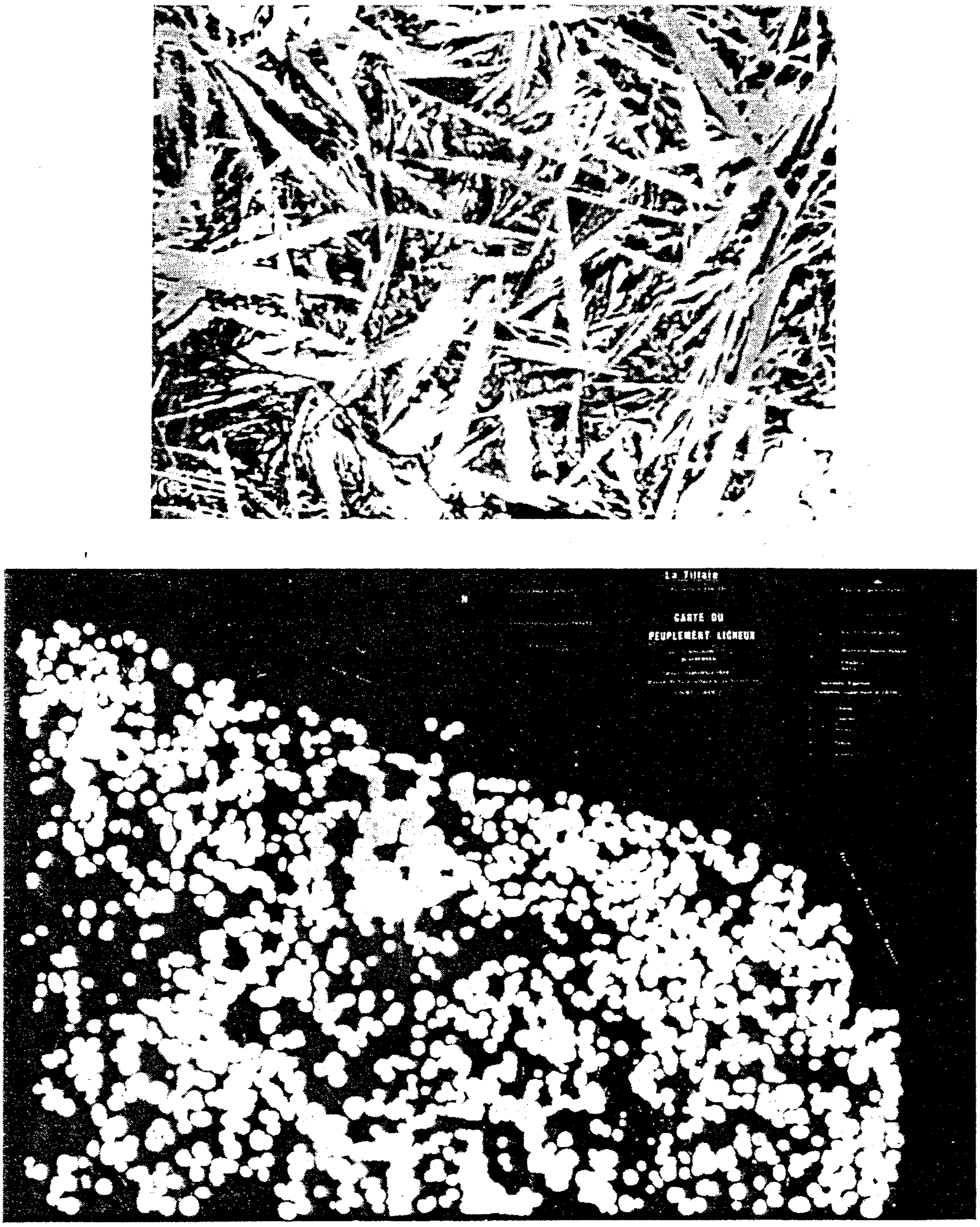
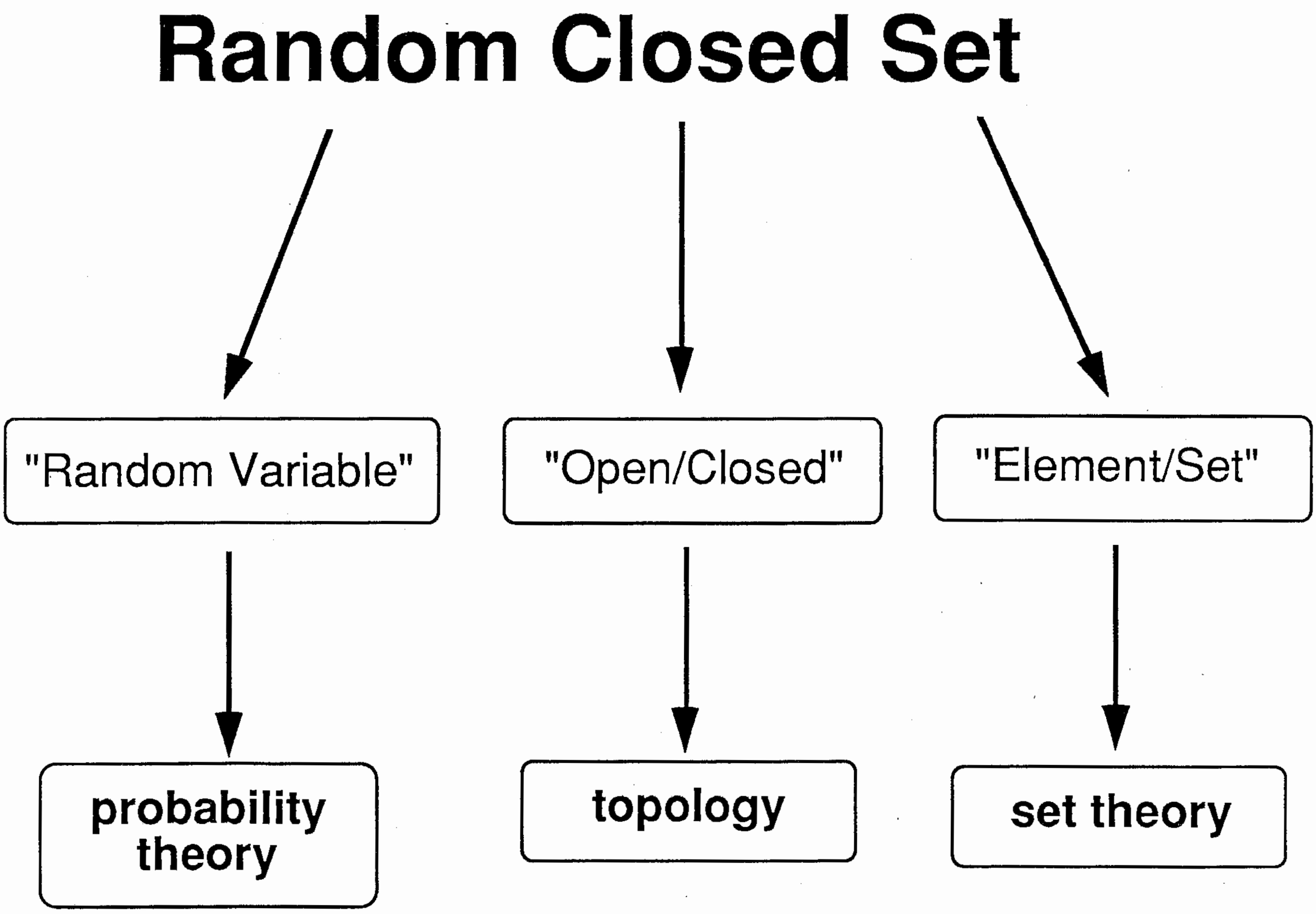


Figure XIII.2. Two examples of phenomena suitably modelled by Boolean sets. (a) (above) ferrite crystals in an iron sinter, (b) (below) portion of the forest of Fontainebleau. (See also micrographs of clay Figure V.31 (× 3.000) and oogoniae Figure IX-13(a), which are both Boolean. In the second example, the primary grain is made of two disjoint cells.)

From reference 13

I. What is a Random Closed Set?
確率閉集合とは？



1. Review of topology

A **topological space** is a pair (U,Ω), where U is a set ("universe") and Ω is a set of subsets of U that verifies:

i) $\varnothing \in \Omega$, $U \in \Omega$

ii) if $O_1 \in \Omega$ and $O_2 \in \Omega$, then $O_1 \cap O_2 \in \Omega$

(i.e. Ω is stable for finite intersection)

iii) if $O_\lambda \in \Omega$, then $\bigcup_{\lambda \in \Lambda} O_\lambda \in \Omega$

(i.e. Ω is stable for union)

Elements of Ω are called ***open subsets*** of U, their complements in U ***closed subsets*** of U.

Examples of topological spaces

1) **Discrete topology**

   $(U,P(U))$ is a topological space for **any** set U.

   $\Rightarrow$ every subset is **both** open and closed!

2) **Trivial topology**

   $(U,\{\varnothing,U\})$ is a topological space for **any** set U.

   $\Rightarrow$ only $\varnothing$ and U are open (and closed)!

3) **Metric topology**

   if (U,δ) is a metric space (distance δ), (U,Ω) where Ω is generated by the ***open balls*** is a topological space.

   $\mathring{B}(O,\rho)=\{x\in U,\delta(x,O)<\rho\}$ open ball of center O and radius ρ

4) **Euclidean topology** $(\Re^n,\delta)$

   = metric topology in $\Re^n$ with δ the euclidean distance:

   $$\delta(x(x_1,\ldots,x_n),y(y_1,\ldots,y_n))=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

Continuity of a mapping

f: $(U_1, \Omega_1) \mapsto (U_2, \Omega_2)$ mapping from topological space (U_1, Ω_1)

$x \to y$ to topological space (U_2, Ω_2)

f is ***continuous*** iff the inverse image of every open set of U_2 is open in U_1:

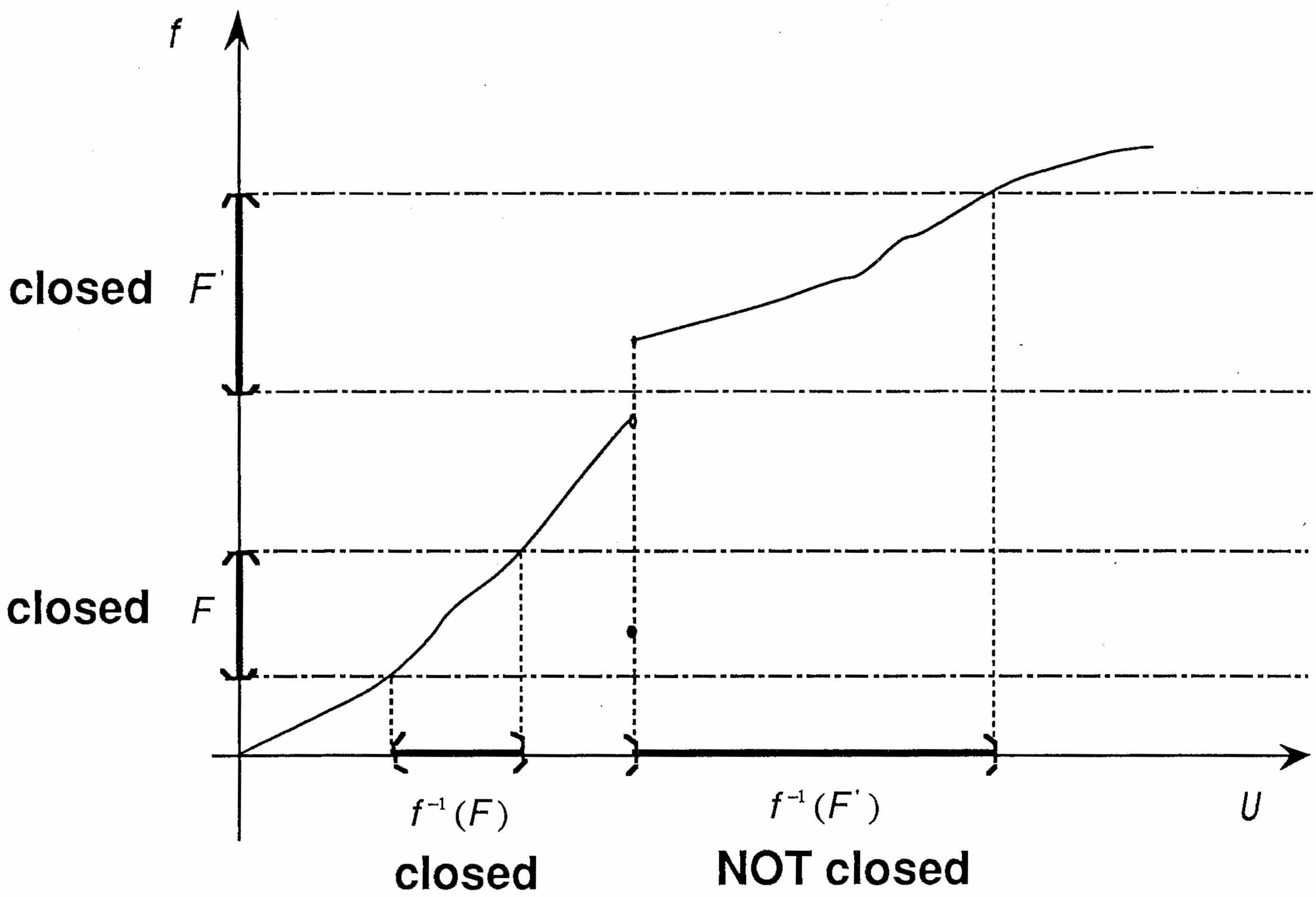
$$\forall O_2 \in \Omega_2, f^{-1}(O_2) \in \Omega_1$$

$$\Updownarrow$$

f is ***continuous*** iff the inverse image of every closed set of U_2 is closed in U_1:

$$\forall F_2 \in \Phi_2, f^{-1}(F_2) \in \Phi_1$$

continuity, discontinuity



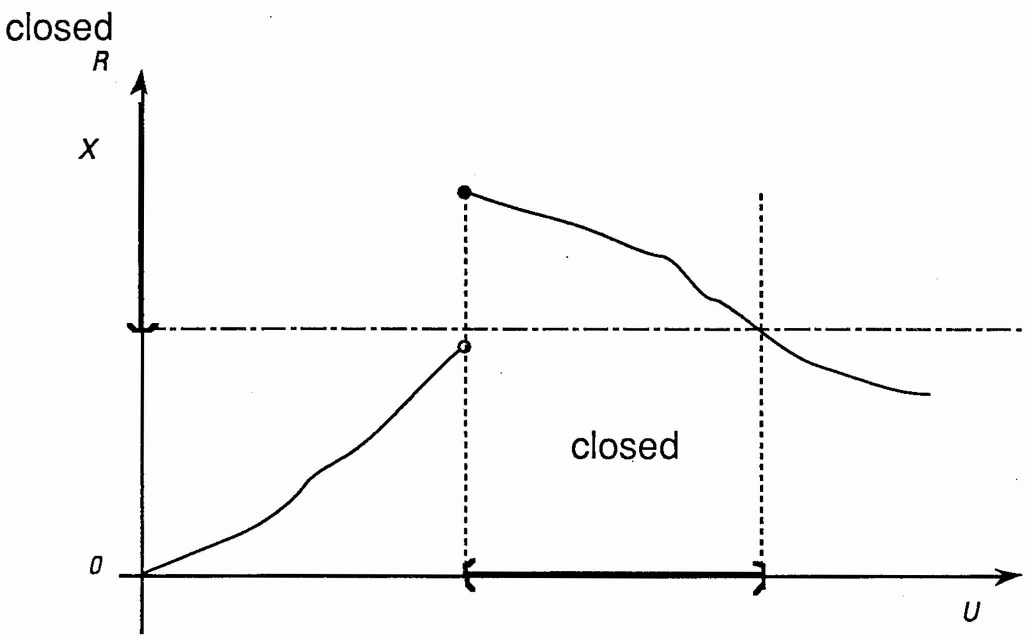
Semi-continuity of a function

$f:(U,\Omega)\mapsto(\Re,\Omega_\delta)$ — function from topological space (U,Ω)

$x\rightarrow y$ — to metric topological space $(\Re,\Omega_\delta)$

f is ***upper semi - continuous*** iff the inverse image of every closed set $[x,+\infty[$ of $\Re$ is closed in U: $\forall x\in\Re, f^{-1}([x,+\infty[)\in\Phi$

f is ***lower semi - continuous*** iff the inverse image of every closed set $]-\infty,x]$ of $\Re$ is closed in U: $\forall x\in\Re, f^{-1}(]-\infty,x])\in\Phi$



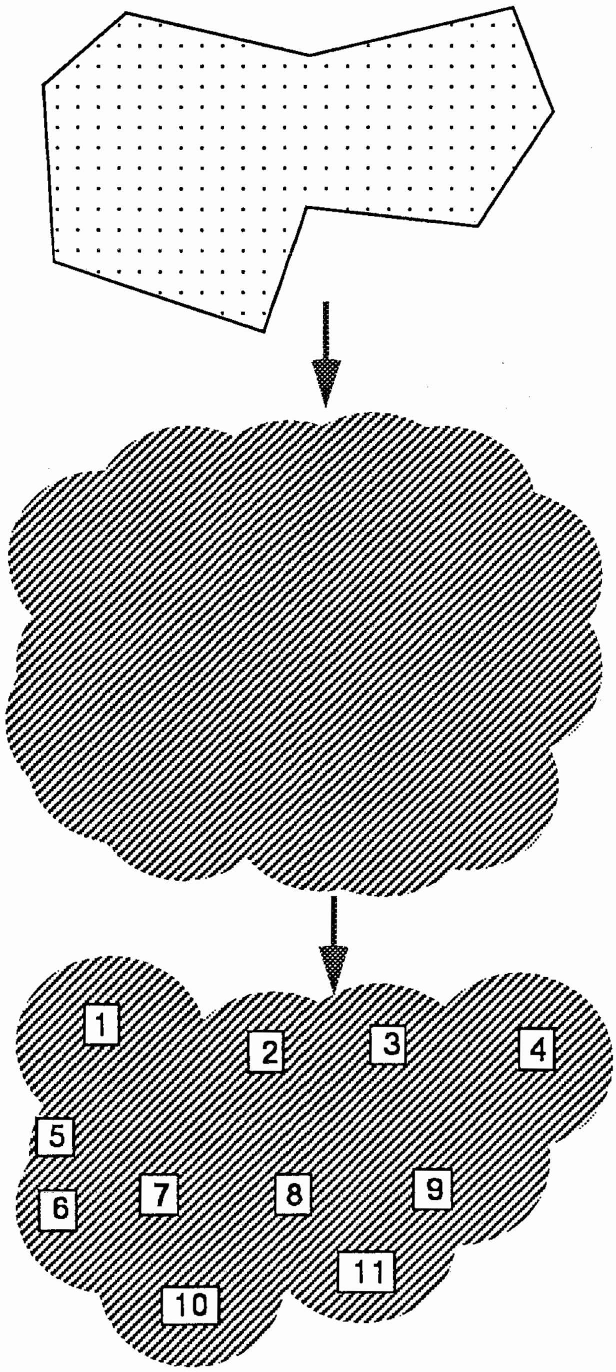
Compact sets

A subset K of a topological space (U,Ω) is ***compact*** iff from any covering of K by opens, we can extract a finite subcovering.

compact set

covering by an infinite number of open balls

finite subcovering



$\Longrightarrow$ Compact sets are always closed.

In metric spaces, compact sets = bounded closed sets

2. Review of probability measure theory

A σ-***algebra*** (σ-**field**) Σ of U is a set of subsets of U stable for countable unions and complementation:

i) $\forall A \in \Sigma, A^c \in \Sigma$

ii) $\forall A_1, \ldots, A_i, \ldots \in \Sigma, \bigcup_{i=1}^{+\infty} A_i \in \Sigma$

The elements of a σ-algebra are called ***events***.

(U, Σ) is called a ***measurable space***.

A function $\mu : \Sigma \mapsto \Re$ is a ***measure*** iff:

i) $\forall A \in \Sigma, 0 \leq \mu(A) \leq \infty$

ii) $\mu(\varnothing) = 0$

iii) $\forall A_i \in \Sigma$ (for $i = 1, 2, \ldots$), $A_i \cap A_j = \varnothing$ whenever $i \neq j$,

then $\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i)$

(U, Σ, μ) is called a ***measure space***.

If $\mu(U) = 1$ then μ is called a ***probability measure*** and (U, Σ, μ) a ***probability space***.

Measurable mappings
Random variables

$f:(U_1,\Sigma_1) \mapsto (U_2,\Sigma_2)$ mapping from a measurable space into another is said ***measurable*** iff the inverse image $f^{-1}(A_2)$ of every event A_2 in U_2 is an event in U_1: $\forall A_2 \in \Sigma_2, f^{-1}(A_2) \in \Sigma_1$.

A measurable mapping $f:(U_1,\Sigma_1,\Pr) \mapsto (U_2,\Sigma_2)$ from a probability space into a measurable space is called a ***random variable***.

f induces a probability measure P in (U_2,Σ_2):

$$P(A_2) = \Pr\left(f^{-1}(A_2)\right) = \Pr\left(\{x \in U_1, f(x) \in A_2\}\right)$$

noted $\Pr\left(f(x) \in A_2\right)$ or even $\Pr\left(f \in A_2\right)$

Example of random variable

Let f: $(U,\Sigma,\Pr) \mapsto (\Re,\Sigma_{\Re})$ be a real - valued random variable.

$\Sigma_{\Re}$ is generated by the open intervals (balls): $]a,b[$

$$
\begin{aligned}
P(]a,b[) &= \Pr\left(f^{-1}(]a,b[)\right) = \Pr\left(\{x \in U, f(x) \in]a,b[\}\right) \\
&= \Pr\left(\{x \in U, a < f(x) < b\}\right) \\
&\text{noted } \Pr(a < f < b)
\end{aligned}
$$

Topological σ-algebras and measures

If (U,Ω) is a topological space, the smallest σ-algebra Σ **containing all the open sets** $(\Sigma \supset \Omega)$ is called the ***Borel*** σ***-algebra*** of (U,Ω) and measures defined on Σ are called ***Borel measures***.

Σ contains all the open sets, all the closed sets and all countable intersections and unions of open and closed sets.

3. Random Closed Sets

3.1 Hit or Miss topology

Let (U,Ω) be a topological space.

We want to build a *topological structure* on the set Φ of the **closed subsets** of U.

Consider the sets

$O_K^O = \{F \in \Phi, F \text{ hits } O, F \text{ misses } K\} = \{F \in \Phi, F \cap O \neq \varnothing, F \cap K = \varnothing\}$

for all opens O and all compacts K in U.

The topology generated by all O_K^O is called the ***Hit or Miss topology***.

3.2 definition of Random Closed Sets

A ***Random Closed Set*** (RACS) is a random variable valued in the measurable space (Φ,Σ_Θ), where Σ_Θ is the Borel σ-algebra of (Φ,Θ).

$X:(U_0,\Sigma_0,\Pr)\mapsto(\Phi,\Sigma_\Theta,P)$

$P(A)=\Pr(X\in A)$ for A event in Σ_Θ

Since Σ_Θ is generated by O_K^O, it is sufficient to work with the O_K^O:

$P(O_K^O)=\Pr(X\in O_K^O)=\Pr(X \text{ hits } O \text{ and } X \text{ misses } K)$

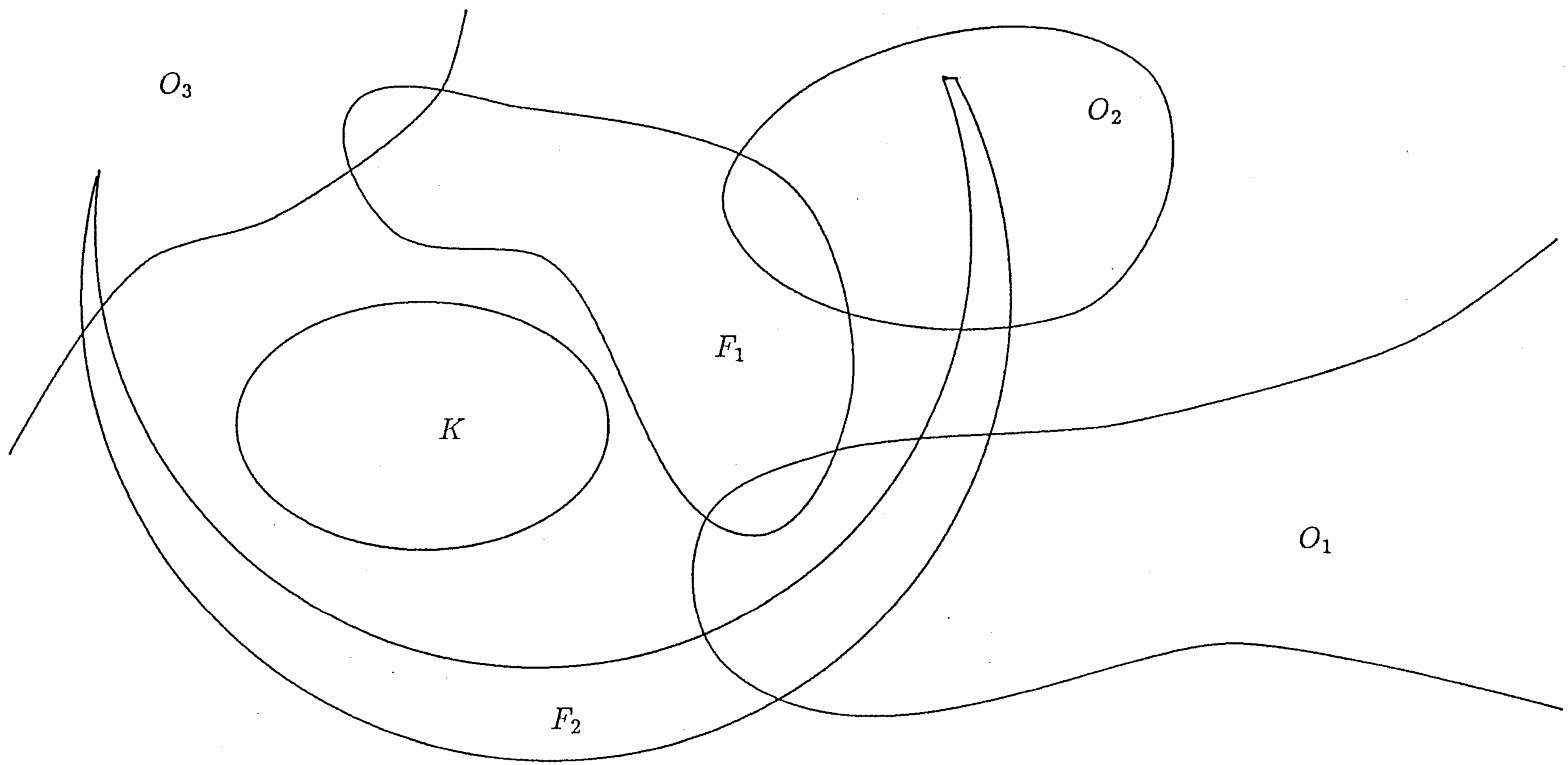


Fig.2: Hit or Miss topology (F_1 and F_2 belong to the same neighborhood $O'_{O_1} \cap O'_{O_2} \cap O'_{O_3} \cap O'^{K}$)

Choquet's theorem

A RACS X is entirely determined by its hitting functional T_X defined on the compacts of U: $T_X(K) = \Pr(X \text{ hits } K) = \Pr(X \cap K \neq \emptyset)$.

Conversely, a functional T defines a unique RACS X iff:

i) $T(K) \leq 1$

ii) $T(\emptyset) = 0$

iii) $T(K_n) \downarrow T(K)$ whenever $K_n \downarrow K$ (sequential continuity)

iv) $\forall n > 0, \forall (K_0, K_1, \ldots, K_n), T(K_0) \leq \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|+1} T\left(K_0 \cup \bigcup_{i \in I} K_i\right)$

T_X is an alternating Choquet capacity of infinite order, called ***hitting capacity*** of X.

RACS in finite spaces

Finite sets can be made *topological spaces*: discrete topology

=> all subsets are open, closed and compact

=> Random Closed Sets are simply called ***random sets.***

Summary of part I

A **Random Closed Set** is simply a random variable valued in a set of subsets, instead of a set of points.

These subsets are **closed** in a topological sense.

A RACS is entirely determined by knowing the probability that it **hits** some given compacts.

These compacts are **structuring elements** that we use to *probe* our random set.

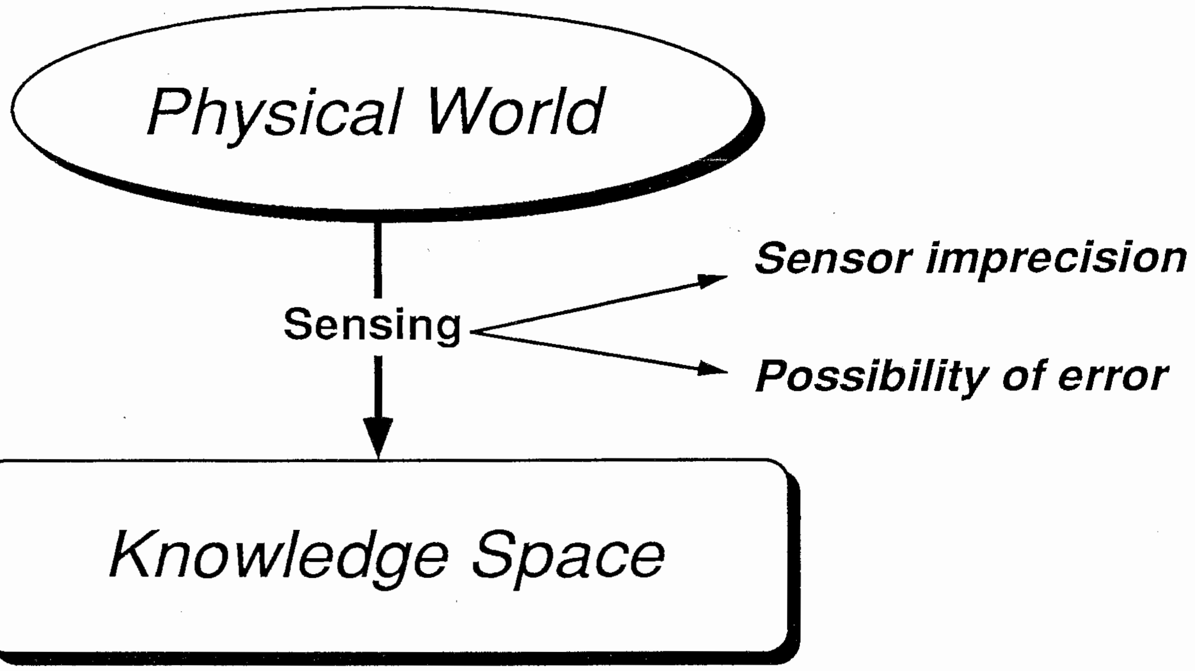
II. Links with other theories of Uncertainty
従来のあいまいさ理論との関係

1. **a semantics for Random Closed Sets**
   - 1.1 the knowledge acquisition problem: Imprecision vs. Uncertainty
   - 1.2 random sets
   - 1.3 experimental accessibility and topology
   - 1.4 Random Closed Sets

2. **link with the Dempster-Shafer theory**
   - 2.1 Belief and Plausibility functions
   - 2.2 relation to hitting and inclusion capacities
   - 2.3 summary of the links
   - 2.4 extensions of the Dempster-Shafer theory based on RACS

3. **link with the Fuzzy set theory**
   - 3.1 Fuzzy set membership functions
   - 3.2 Fuzzy connectives

Knowledge Acquisition



The acquired knowledge is inherently *imprecise* and possibly *erroneous*, hence *uncertain*

↓

Explicit representation of **both *imprecision* and *uncertainty*** is required

Random Sets

1) *Imprecision* is ***set-theoretic***
   i.e. mathematically described by *Set theory*.

   Ex: error intervals in physics
   "temperature $\boldsymbol{\theta}$ is 20°±1°" means "$\theta \in [19°, 21°]$"

2) *Uncertainty* is ***probabilistic***
   i.e. mathematically described by *Probability Measure theory*.

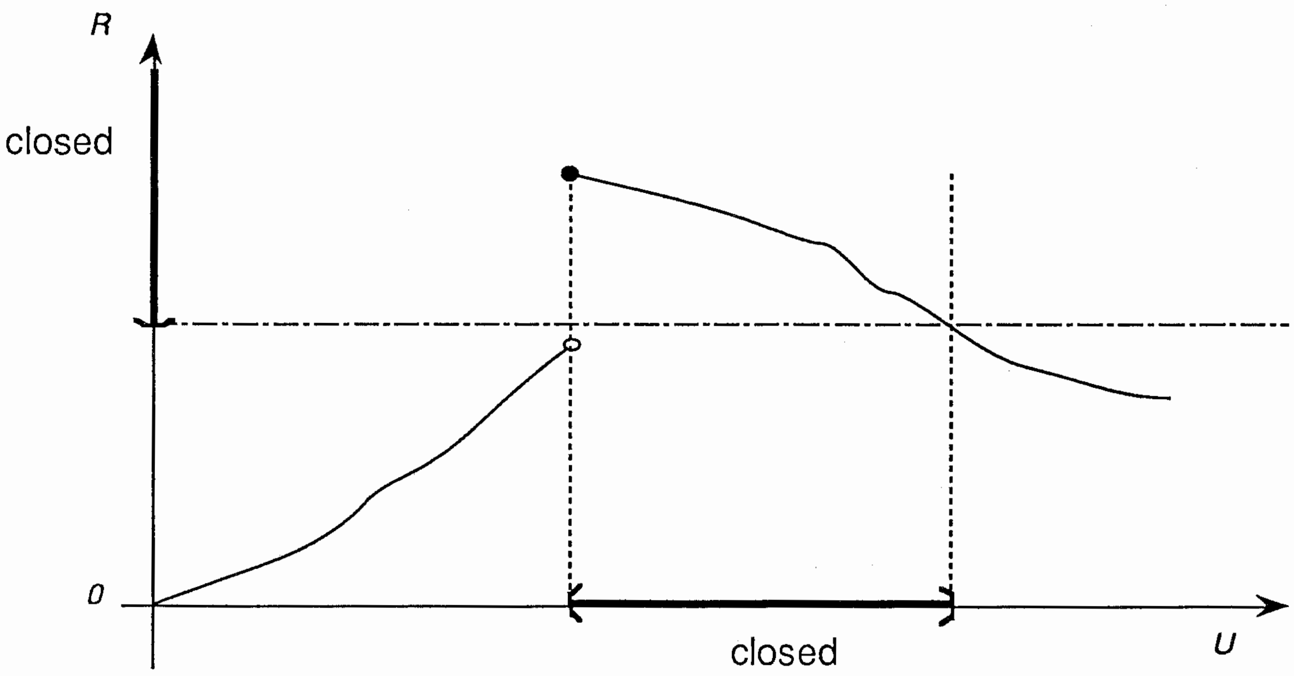
   Ex: random point variables valued in the real line
   "it is **50% certain** that temperature $\boldsymbol{\theta}$ is 20°±1°"
   means "$\Pr(\theta \in [19°, 21°]) = 0.5$"

   Note: Probability Measure theory is **NOT** restricted to random point variables in simple spaces.

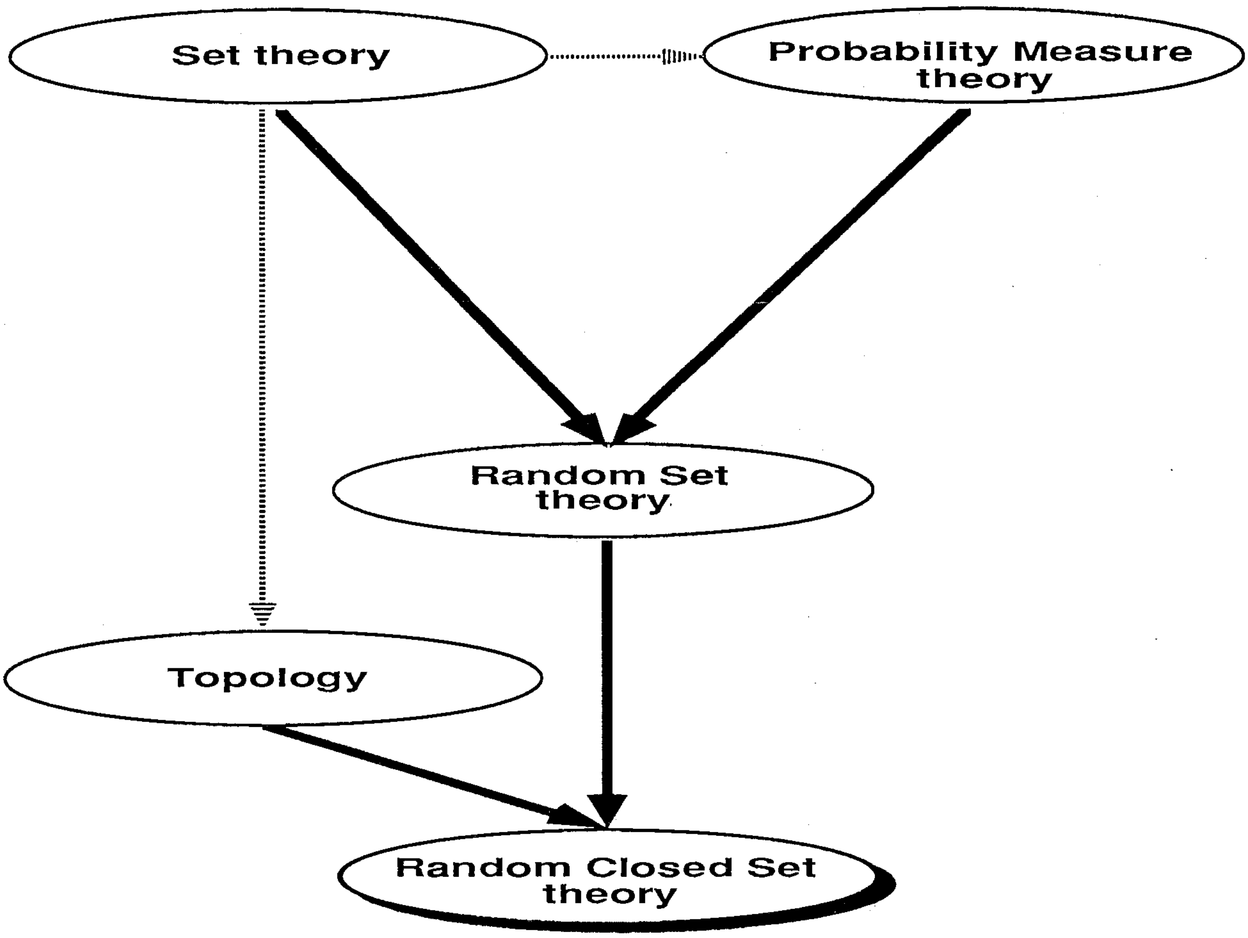
Since we deal with both imprecision **and** uncertainty,
we must use *random set variables*
instead of *random point variables*.

Experimental accessibility and topology

In uncountably infinite spaces, ***experimental accessibility*** implies a notion of ***topology*** ("neighborhood", "continuity"...)



Random Closed Sets (RACS)



<u>Choquet's theorem</u>:

A random closed set X is entirely determined by its ***hitting (Choquet) capacity*** :

$$T_X(K) = \Pr(X \,\mathrm{hits}\, K)$$

Relation to the Dempster-Shafer theory
(in a finite space)

Belief function Bel:

$$\mathrm{Bel}:\ P(U) \mapsto [0,1]$$
$$A \to \mathrm{Bel}(A)$$

(i) $\mathrm{Bel}(\varnothing) = 0$

(ii) $\mathrm{Bel}(U) = 1$

(iii) $\mathrm{Bel}\left(\bigcup_{i=1,\dots,n} A_i\right) \geq \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \varnothing}} (-1)^{|I|+1}\, \mathrm{Bel}\left(\bigcap_{i \in I} A_i\right)$ for every finite family $A_1, \dots, A_n$

Plausibility function Pl:

$$\mathrm{Pl}:\ P(U) \mapsto [0,1]$$
$$A \to \mathrm{Pl}(A)$$

(i) $\mathrm{Pl}(\varnothing) = 0$

(ii) $\mathrm{Pl}(U) = 1$

(iii) $\mathrm{Pl}\left(\bigcap_{i=1,\dots,n} A_i\right) \leq \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \varnothing}} (-1)^{|I|+1}\, \mathrm{Pl}\left(\bigcup_{i \in I} A_i\right)$ for every finite family $A_1, \dots, A_n$

Belief interval of A:

$$[\mathrm{Bel}(A), \mathrm{Pl}(A)] = [\mathrm{Bel}(A), 1 - \mathrm{Bel}(A^c)] = [1 - \mathrm{Pl}(A^c), \mathrm{Pl}(A)] \subset [0,1]$$

Ignorance with respect to A: $\mathrm{Pl}(A) - \mathrm{Bel}(A)$

Relation to the Dempster-Shafer theory
(in a finite space)

U: finite set --> topological space with discrete topology

--> "discrete topological space"

The ***Belief functions*** *Bel* defined on a (finite) universe *U* are exactly the ***inclusion capacities*** of the <u>almost surely non-empty</u> Random (Closed) Sets *X* of the discrete topological space *U.*

The ***Plausibility functions*** *Pl* defined on a (finite) universe *U* are exactly the ***hitting capacities*** of the <u>almost surely non-empty</u> Random (Closed) Sets *X* of the discrete topological space *U.*

Relation to the Dempster-Shafer theory
(in a finite space)

Dempster-Shafer		***RACS***
$\mathrm{Pl}(K)$ **Plausibility** function	⟷	$T_X(K) = \mathrm{Pr}(X \text{ hits } K)$ **Hitting capacity** of a.s. non-empty RACS
$\mathrm{Bel}(K)$ **Belief** function	⟷	$P_X(K) = \mathrm{Pr}(X \subset K)$ **Inclusion capacity** of a.s. non-empty RACS
$\mathrm{q}(K)$ **Communality** function	⟷	$R_X(K) = \mathrm{Pr}(K \subset X)$ **Implying functional** of a.s. non-empty RACS
$\mathrm{m}(K)$ **Mass** function	⟷	$\mathrm{Pr}(X = K)$ **Probability density** with respect to the counting measure
$[\mathrm{Bel}_1 \oplus \mathrm{Bel}_2](K) = \sum_{A \text{ hits } K} \left(\sum_{A = B \cap C} \mathrm{m}_1(B) \cdot \mathrm{m}_2(C) \right)$ Unnormalized **Dempster's rule**	⟷	$T_{X_1 \cap X_2}(K) = \sum_{A \text{ hits } K} \left(\sum_{A = B \cap C} \mathrm{Pr}(X_1 = B; X_2 = C) \right)$ **Intersection** of independent a.s. non-empty RACS
$[\mathrm{Bel}_1 \oplus \mathrm{Bel}_2](K) = \dfrac{\sum_{A \text{ hits } K} \left(\sum_{A = B \cap C} \mathrm{m}_1(B) \cdot \mathrm{m}_2(C) \right)}{1 - \sum_{B \cap C = \varnothing} \mathrm{m}_1(B) \cdot \mathrm{m}_2(C)}$ Normalized **Dempster's rule**	⟷	$T_{X_1 \oplus X_2}(K) = \dfrac{\sum_{A \text{ hits } K} \left(\sum_{A = B \cap C} \mathrm{Pr}(X_1 = B; X_2 = C) \right)}{1 - \sum_{B \cap C = \varnothing} \mathrm{Pr}(X_1 = B; X_2 = C)}$ **Conditional intersection** of independent a.s. non-empty RACS given that they are not disjoint

therefore...

The Dempster-Shafer theory is the theory of ***statistically independent, almost surely non-empty Random Closed Sets in finite spaces, with intersection*** *as the combination operator*

Extensions of the Dempster-Shafer theory

1) **Open World** assumption (Smets 1988, Matsuyama 1989)

=> drop the "almost surely non-empty" requirement

Theory of ***statistically independent Random Closed Sets in finite spaces, with intersection*** *as the combination operator*

2) extension to **infinite** spaces

=> Random Closed Sets are defined in general compact metrizable topological spaces (Matheron 1975)

Theory of ***statistically independent Random Closed Sets with intersection*** *as the combination operator*

3) Dempster's rule with **non-independent** pieces of evidence

$$T_{X_1 \cap X_2}(K) = \sum_{A \text{ hits } K} \left(\sum_{A = B \cap C} \Pr(X_1 = B; X_2 = C) \right)$$

Theory of ***Random Closed Sets with intersection*** *as the combination operator*

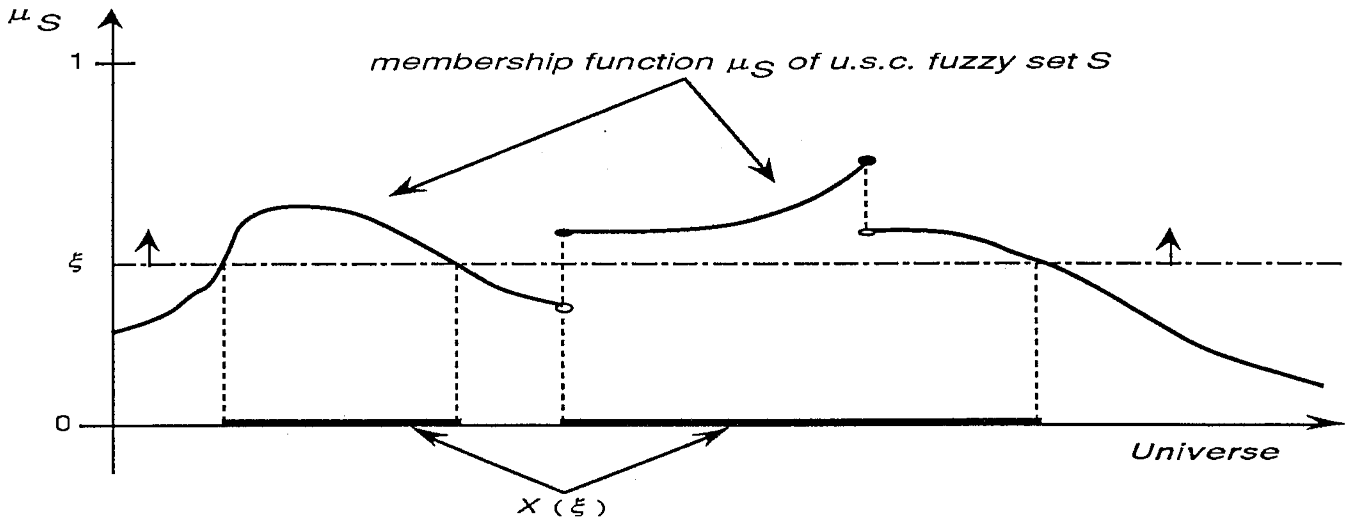
Relation to the Fuzzy set theory

fuzzy set membership functions

fuzzy sets $\underset{\textit{point coverage}}{\overset{\textit{cross-section}}{\rightleftarrows}}$ Random sets

(Goodman 1982)

(upper) **semi-continuous** fuzzy sets $\underset{\textit{point coverage}}{\overset{\textit{cross-section}}{\rightleftarrows}}$ Random **Closed** Sets



Cross section function $\nu(\xi) = \mu_S^{-1}([\xi;1])$

Particularization

Random Closed Sets (RACS) $\Pr(X \,\text{hits}\, K) = T_X(K)$	***Fuzzy sets*** $\Pr(X \,\text{hits}\, \{k\}) = \mu(k)$
Random points $\Pr(\{x\} \,\text{hits}\, K) = \Pr(x \in K)$	***Finite probabilities*** $\Pr(\{x\} \,\text{hits}\, \{k\}) = \Pr(x = k)$

$$\underset{\textbf{closed set}}{A} \xrightarrow{\textit{particularization}} \underset{\textbf{singleton}}{\{a\}}$$

Relation to the Fuzzy set theory

fuzzy connectives

Fuzzy sets | *Random Closed Sets*

$$
\begin{tikzcd}
S_1 \arrow[rr, "\kappa"] \arrow[d] & & X_1 \arrow[d] \\
S_1 \wedge S_2 & & X_1 \cap X_2 \arrow[ll, "\varphi"'] \\
S_2 \arrow[rr, "\kappa"'] \arrow[u] & & X_2 \arrow[u]
\end{tikzcd}
$$

For any T-norm $\wedge$, any S_1 and S_2, there exist 2 RACS X_1 and X_2 such that: $S_1 = \varphi(X_1)$, $S_2 = \varphi(X_2)$, $S_1 \wedge S_2 = \varphi(X_1 \cap X_2)$

Relation to the Fuzzy set theory

fuzzy connectives

The binary operator $\wedge$ defined by: $S_1 \wedge S_2 = \varphi\left(\kappa(S_1) \cap \kappa(S_2)\right)$ is **commutative**, **associative** and verifies the **boundary conditions** of a T-norm, but is not necessarily **monotonic**. Several operators can be obtained, some of which are T-norms, depending on the statistical dependence between $\kappa(S_1)$ and $\kappa(S_2)$.

Upper/Lower probabilities induced by a set of probability measures
$\Leftrightarrow$ Generalized Possibility/Necessity theory

Upper/Lower probabilities induced by a multivalued mapping
$\Leftrightarrow$ Random Closed Set (RACS) theory

Almost surely non-empty Random Closed Sets
$\Leftrightarrow$ Dempster-Shafer theory (Belief/Plausibility)

Upper semi-continuous Fuzzy sets

Zadeh Possibility/Necessity measures

therefore...

1) **Both** the Dempster-Shafer and the Fuzzy set theories can be expressed within the ***purely probabilistic*** framework of Random Closed Sets

2) the widely used combination operators are all essentially **equivalent** to ***set-theoretic*** operators ($\cap$ for Dempster's rule, $\cap$ or $\cup$ for the fuzzy connectives)

*These combination operators are suitable for combining **imprecise but certain** knowledge, but inadequate when knowledge is **both imprecise and uncertain.***

*To combine imprecise **and** uncertain knowledge, a **probabilistic operator** is required...*

III. Implementational issues and applications
実用にあたっての問題点

1. statistical estimation of RACS

1.1 parametric vs. non-parametric estimation

1.2 an example of RACS model: the Boolean model

1.3 the choice of structuring elements
in non-parametric estimation

2. application to texture analysis & synthesis

2.1 model fitting with the Boolean model

2.2 example

2.3 fractal images and RACS

3. application to stereo-vision

3.1 the stereo-vision context

3.2 the sources of error in stereo-vision:
imprecision vs. uncertainty

3.3 data RACS and visual field RACS

3.4 an octree of structuring elements
for non-parametric estimation

3.5 results and interpretation

4. potential applications to Artificial Intelligence

Statistical Estimation of RACS

We decide to model a particular problem using a Random Closed Set X

==> we must **estimate** the probability measure of the random variable using the available samples *Xi*.

$$\forall A \text{ event } \in \Sigma_\theta, \quad P(A) = \Pr\left(X^{-1}(A)\right) = \Pr\left(\{\alpha, X(\alpha) \in A\}\right) = \Pr(X \in A)$$

Choquet theorem: estimation of the hitting capacity T is enough.

$$T_x(K) = \Pr(X \text{ hits } K)$$

Parametric estimation vs. **Non-parametric estimation**

Model fitting:

stimation of a finite number of model parameters

"Parzen window" method ?

=> only in ***metric*** spaces

An example of a RACS model: **the Boolean model**

Construction: U is a metric vector space (e.g. Euclidean space)

1) X' : Random Closed Set centered at 0 ($\rightarrow$ non-stationary) : **"primary grain"**

2) $\{x_i\}_{i\in I}$: Poisson point process with density θ

$$\rightarrow \begin{cases} \text{- } K_1 \cap K_2 = \varnothing \Rightarrow N(K_1) \text{ and } N(K_2) \text{ statistically } \textbf{independent} \\ \text{- elementary volume } dv \text{ contains 1 point with probability } \theta(dv) \end{cases}$$

We *implant* one realization of X' at each x_i and take the union:

$$X = \bigcup_{i\in I} X'_{x_i}$$

Properties: in general, we have:

$$\begin{cases} Q_X(K) = \Pr(X \text{ misses } K) = \exp\left(-\theta.E\left(\mu(X'\oplus K)\right)\right) \\ T_X(K) = \Pr(X \text{ hits } K) = 1-\exp\left(-\theta.E\left(\mu(X'\oplus K)\right)\right) \end{cases}$$

where μ is the Borel measure of $\Re^n$

The measure of the dilation by a ***convex*** K is given by Steiner's formula in terms of the Minkowsky functionals of $\Re^n$ (see ref.13, pg.111):

$$\text{In } \Re^2 : \; T_X(\lambda.K) = 1-\exp\left(-\theta.\begin{bmatrix} E\left(A(X')\right)+\dfrac{\lambda}{2\pi}U(K).E\left(U(X')\right) \\ +\lambda^2.A(K) \end{bmatrix}\right)$$

where $A(\ )$ and $U(\ )$ are the area and the perimeter in $\Re^2$.

$$\text{In } \Re^3 : \; T_X(\lambda.K) = 1-\exp\left(-\theta.\begin{bmatrix} E\left(V(X')\right)+\dfrac{\lambda}{4\pi}M(K).E\left(S(X')\right) \\ +\dfrac{\lambda}{4\pi}^2.S(K).E\left(M(X')\right)+\lambda^3.V(K) \end{bmatrix}\right)$$

where $V(\ )$, $S(\ )$ and $M(\ )$ are the volume, the surface and the norm in $\Re^3$ ($\leftarrow$ Minkowsky functionals in $\Re^3$).

The choice of structuring elements in non-parametric estimation

If we have NO metrics and NO model, standard non-parametric estimation methods do not apply. **What can we do?**

=> we must select a finite family of compact sets K (**structuring elements**) and probe our sample data with K.

We cannot estimate the hitting capacity functional itself but only **its value** at our structuring elements. This may be sufficient in many practical cases if we do not need the value of T for any arbitrary compact K.

=> we do not define a *unique* RACS anymore, but a family of RACS constrained to behave similarly on the given family of structuring elements.

Application to Texture Analysis/Synthesis

Given a texture pattern, we can try to **fit a RACS model**, estimate the model parameters, store the estimated parameters in memory, and then synthesize the texture from the stored parameters.

<u>Example with the *Boolean model*:</u>

the parameters of the model are: the **density** of the Poisson point process and the information about the **average grain** (expectancy of the Minkowsky functionals of the grain).

From :

pg. 500 IMAGE ANALYSIS AND MATHEMATICAL MORPHOLOGY (reference 13)

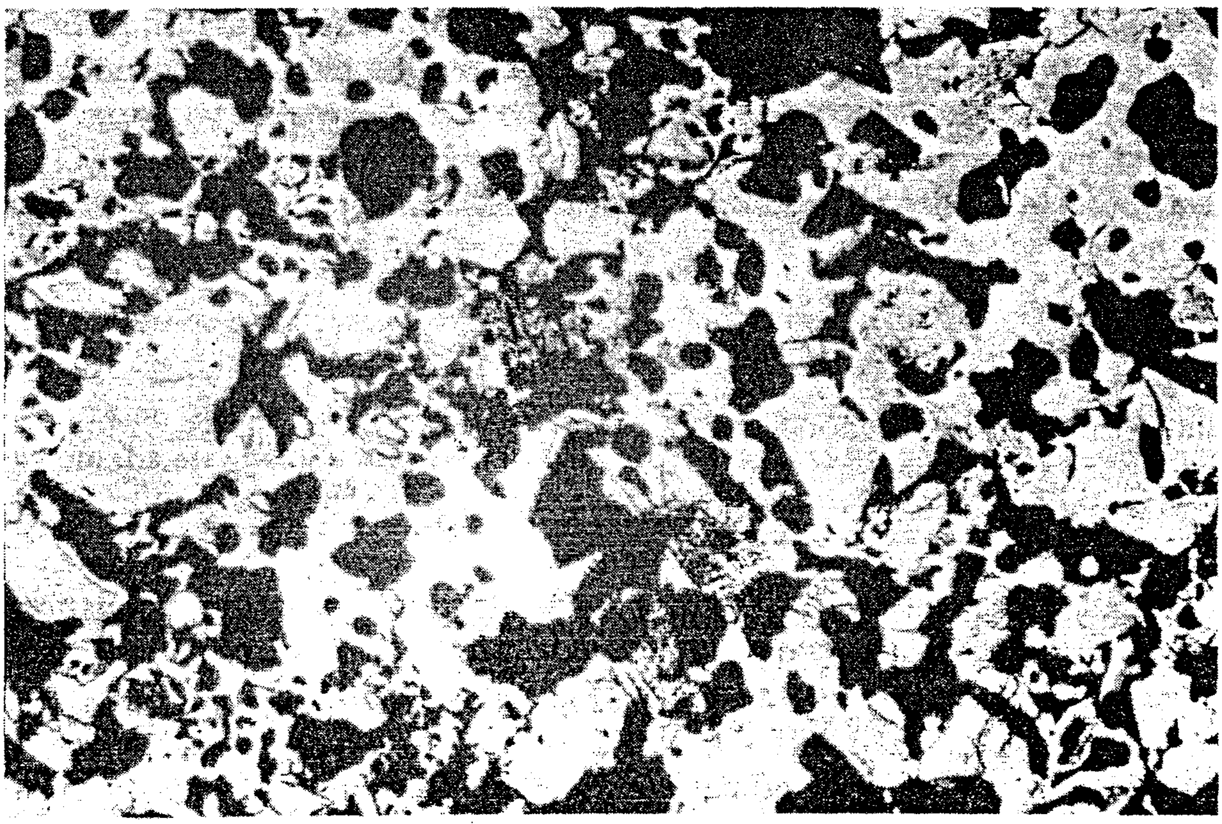
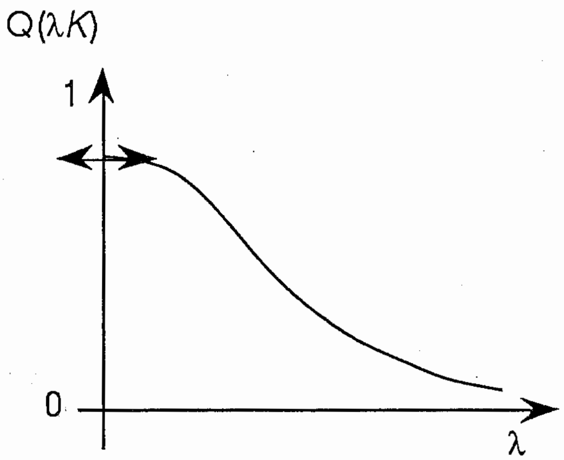


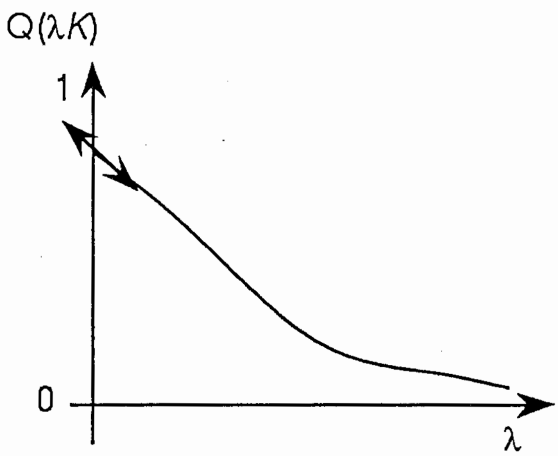
Figure XIII.6. Micrograph from a polished section of formcoke. The carbon appears as white and the pores as dark. The latter will be considered to be the grains when modelled by a Boolean set.

Fractal images and RACS

If an image is well modelled by a RACS X, the behavior of $T_x(\lambda K)$ when $\lambda \to 0$ provides information about the fractal nature of the image, and its fractal dimension.

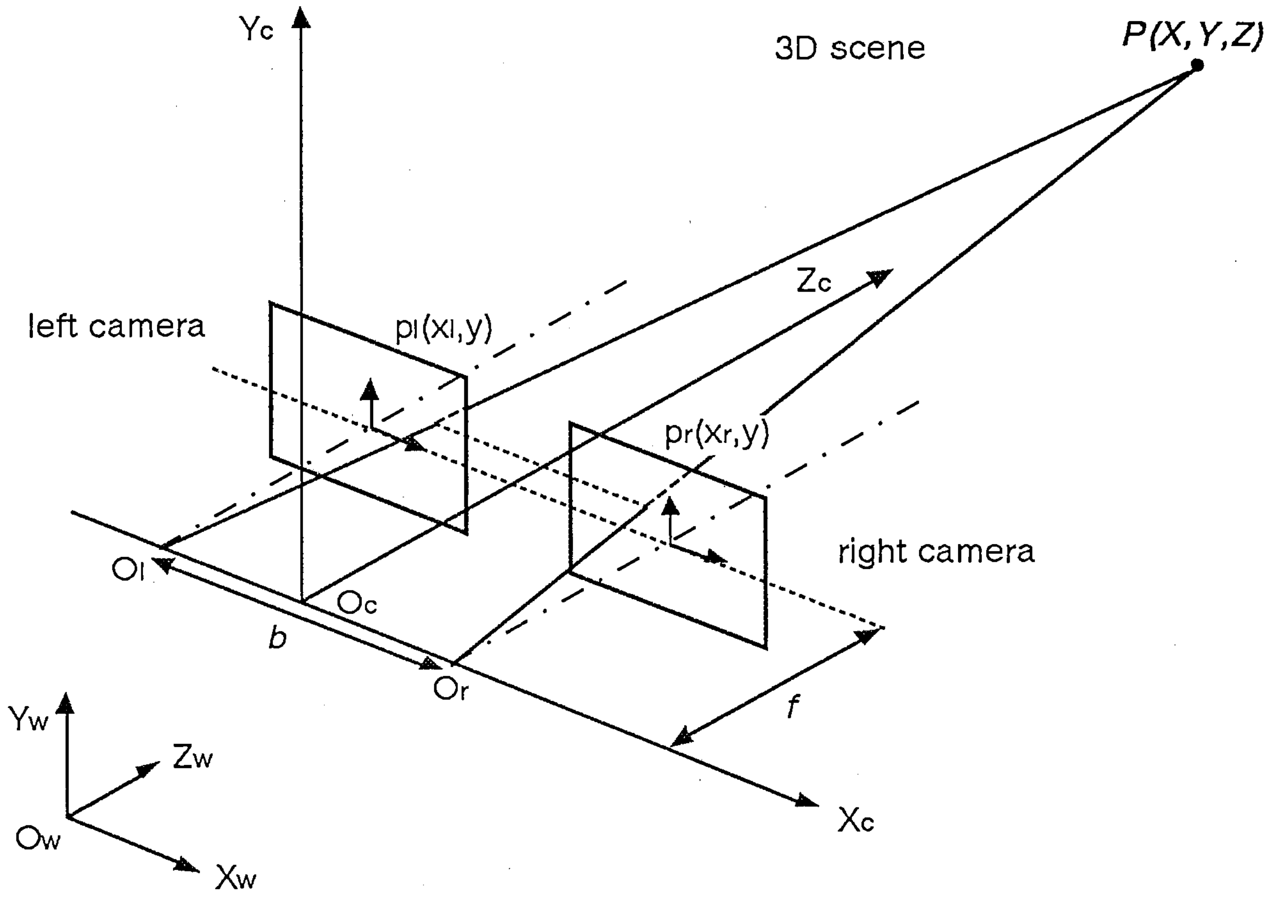


Germ model



Fractal image

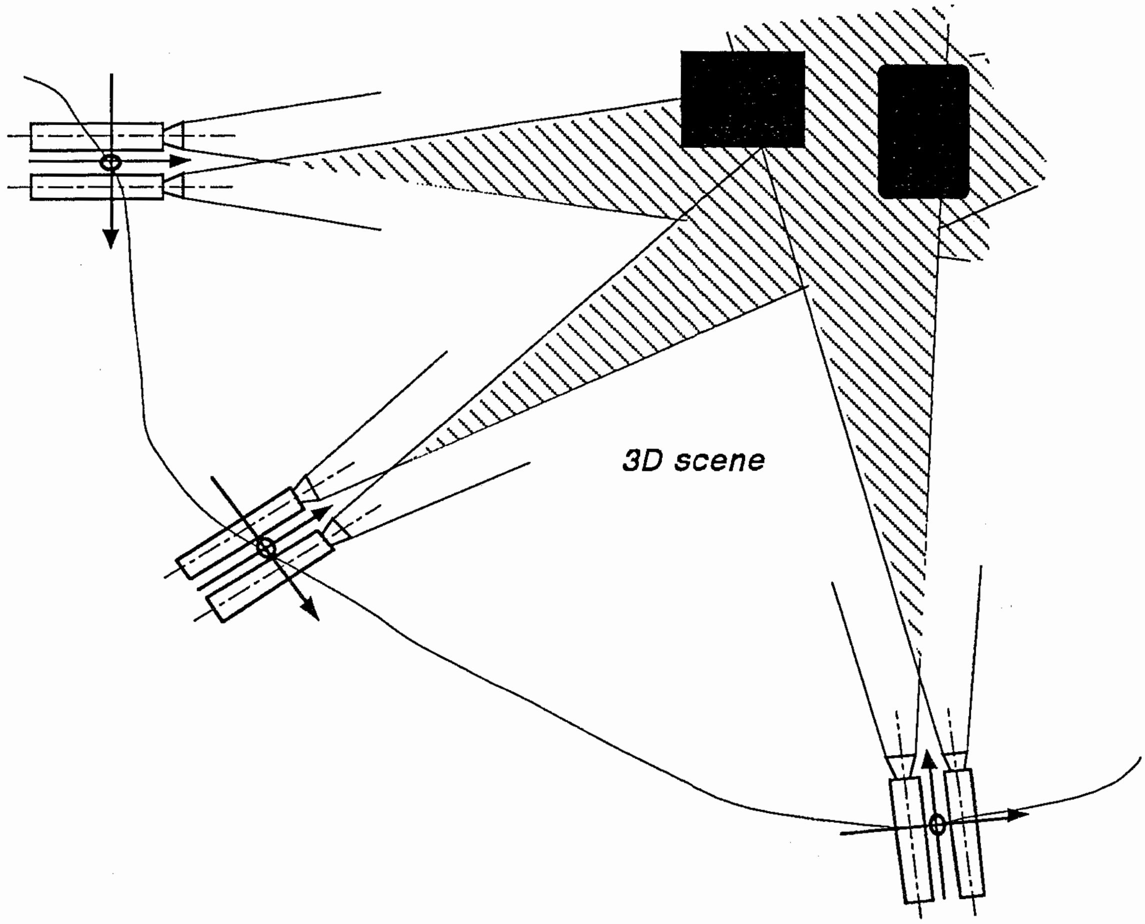
Application to Stereo-vision



Problems: the resulting 3D map is

- **inaccurate**, especially along Zc axis (ΔZc~10cm to 1m for typical values)
- **noisy** (matching errors)
- highly **sensitive** to calibration errors
- **incomplete** (only partial information due to limited visual field, occlusions)

One solution: integration of multiple stereo-pairs

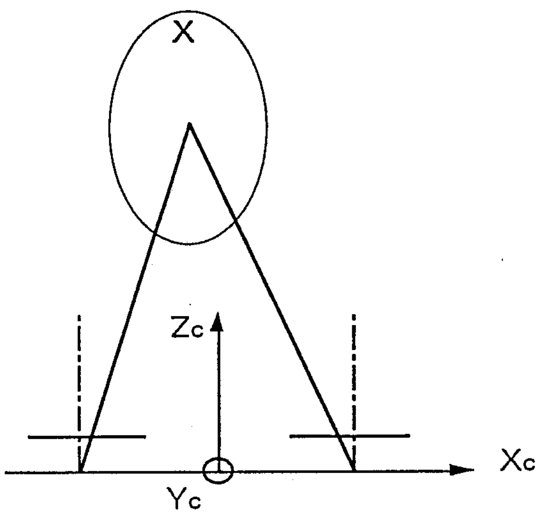


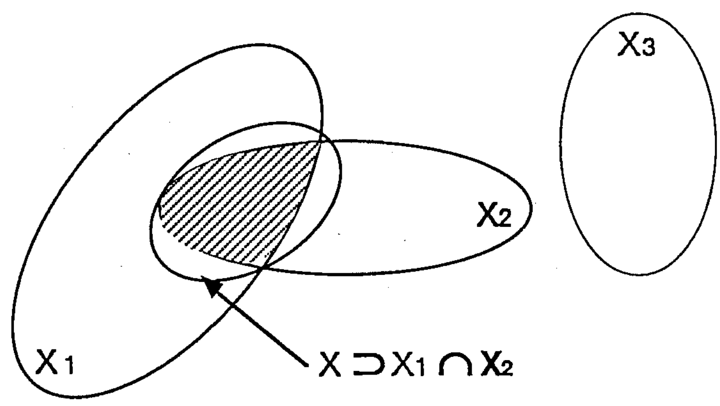
- **redundancy** of 3D information
  - => accuracy increases
  - => errors decrease

- **complementarity** of 3D information
  - => disambiguation (occlusion)
  - => completeness increases

Previous integration methods

1) Model the errors by ***ellipsoids*** and compute an approximation of their ***intersection.***





Problem: what if the ellipsoids do **not** intersect?
should we discard the "bad" data?
which ones are "bad"?

Previous integration methods

2) Model the errors by ***gaussian probability distributions*** in space and compute the "optimal" fused distribution using **Kalman filtering**.

Problem: optimality only guaranteed for **gaussian** distributions.
But errors in 3D space are NOT gaussian!

Ex.: disparity d=0 -> imprecision region in space extends to infinity and actual 3D point may lie anywhere in this region.

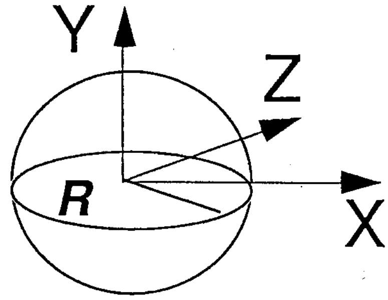
The sources of error:
Imprecision vs. Uncertainty

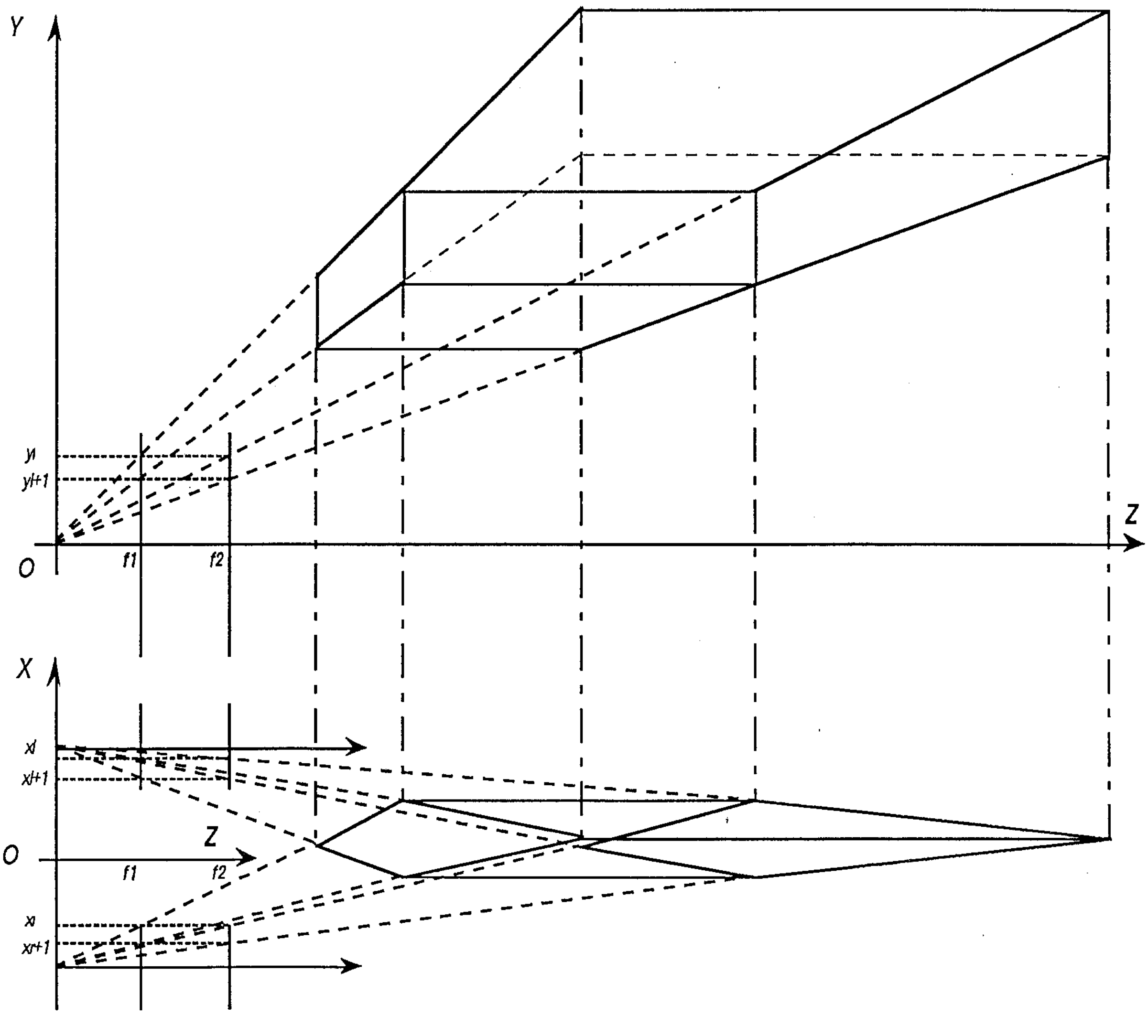
1) **Imprecision** due to:

- digitization (pixels)
- imprecision in system geometry

$$f_1 \le f \le f_2 \qquad b_1 \le b \le b_2$$

-imprecision in system location





The sources of error:
Imprecision vs. Uncertainty

2) **Uncertainty** due to:

random sensor noise

=> erroneous extraction of feature points
=> noise in 3D map

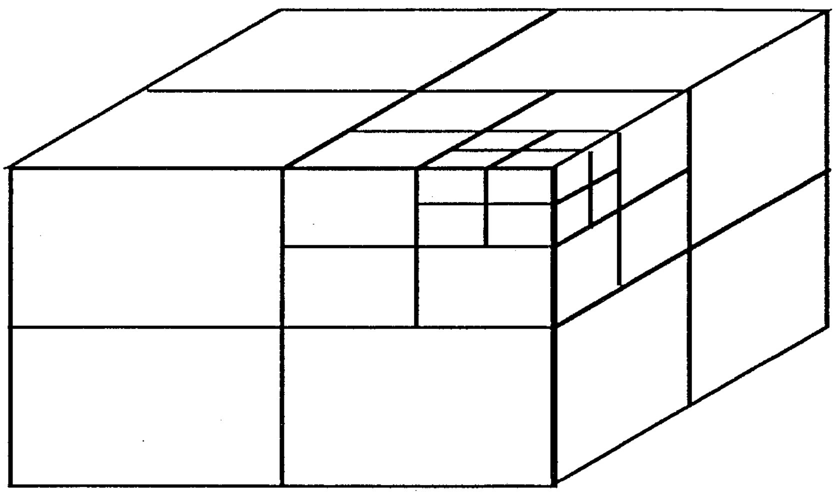
possibility of *matching errors*

when the correlation function is **multimodal**, selecting one mode may lead to errors

=> erroneous 3D information
=> noise in the 3D map

The Structuring Elements

We select a family of **rectangular solids** of different sizes called **OCTREE** (8-tree) as structuring elements.



For every rectangular solid *C* in the tree, we estimate:

$$\text{Prob}(X \text{ hits } C) \approx \frac{\text{nb. of times } X_i \text{ hits } C}{n}$$

Data RACS and Visual Field RACS

The 3D polyhedra created by every disparity vector from image pair *i* form the *Data Random Set* (Data RACS) *Xi*

The 3D polyhedron created by the visual field at location i forms the *Visual Field Random Set* (Visual RACS) *Vi*

What we can easily estimate is:

Prob(X hits C and C is visible)

What we need is:

Prob(X hits C | C is visible)

Statistical Estimation

$$\text{Prob}(X \text{ hits } C \mid C \text{ is visible}) = \frac{\text{Prob}(X \text{ hits } C \text{ and } C \text{ is visible})}{\text{Prob}(C \text{ is visible})}$$

estimated by: $$\boxed{\frac{\text{nb. of } X_i \text{ hitting } C}{\text{nb. of } V_i \supset C}}$$

Indeed, for any given C, the hitting between C and X_i is a Bernouilli random variable with probability of success $\alpha = T_x(C)$ and the number of hittings is the sum of these Bernouilli random variables, hence a binomial random variable if the X_i are independent.
The Central Limit theorem guaranties that the binomial distribution will converge to a normal (Gaussian) distribution of mean $n.\alpha$ as n increases. Hence the average number of hittings is an unbiased estimator of $T_x(C)$.

What to do with the estimated capacity?

Using the estimated hitting capacity functional, we can:

1) use it for **robot navigation**: compute the "best" path in the octree from point A to point B that avoids (misses) scene objects with high probability.

2) use it for **object recognition**: compute a goodness of fit between the octree and objects stored in a database, based on the capacity distribution.

3) **visualize it**: we compare the estimated probability to a threshold τ and keep (show) only the cubes with high hitting probabilities (high plausibilities):

$$\mathrm{Prob}(X \text{ hits } C \mid C \text{ is visible}) > t \qquad (t \in [0,1])$$

- for τ=0, the remaining cubes are the union of all the data sets

=> ***weakest integration***

- for τ->1, the remaining cubes are the intersection of all the data sets

=> ***strongest integration***

Experiments

Example:

Integration of 30 stereo pairs of views of a single object

System parameters:

focal length	1470 pixels $\leq f \leq$ 1520 pixels
baseline	9.95 cm $\leq b \leq$ 10.05 cm

System location and orientation:

location	R = 3mm
orientation	α (yaw) $\pm$ 0.1 deg.
	β (pitch) $\pm$ 0 deg.
	γ (roll) = 0 $\pm$ 0 deg.

See references 10 and 11 for experimental results.

Potential Applications to Artificial Intelligence

U is a space of primitive "objects".

(U,T) is a **topological space**: we can speak of *neighborhoods*, *continuity* in *U*.

A **compact** *K* of *U* is a "concept" in *U*.

We receive **data** = *imprecise and uncertain* pieces of information about the observable world ==> modelled by a RACS *X*.

We are interested in the **plausibility of a concept** *K* in view of the data *X*:

Plausibility of *K* in view of $X = \mathrm{Pr}(X \text{ hits } K) = \mathrm{Pl}(K)$

and the **amount of support (belief) of a concept** K in view of X:

Support (belief) of *K* in view of $X = \mathrm{Pr}(X \subset K) = \mathrm{Bel}(K)$

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